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Stochastic Optimal Growth through State-Dependent Probabilities

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Abstract

We extend the classical discrete time stochastic one-sector optimal growth model with logarithmic utility and Cobb-Douglas production á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. In this setting the probability of occurrence of a given shock depends on the capital stock, thus, as the economy accumulates more capital, the probability of occurrence of different shocks changes over time. We explicitly determine the optimal policy and its relation with state-dependent probabilities both in the centralized and decentralized frameworks, focusing on two alternative scenarios in which the probability function, assumed to take a logarithmic form, is either decreasing or increasing with capital. We show that state-dependent probabilities introduce a wedge between the centralized and decentralized solutions, as individual agents do not internalize the effects of capital accumulation on the probability of shocks realization. In particular, whenever the probability is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, leading the decentralized economy to underinvest (overinvest) in capital accumulation, resulting in the long run into a steady state capital distribution characterized by a leftward (rightward) shifted support. We also show how the features of state-dependent probabilities affect the spread and shape of such a steady state distribution, which tends to be more skewed (more evenly spread) whenever the probability decreases (increases) with capital.

Keywords: Brock and Mirman Model; Iterated Function Systems; Optimal Growth; State-Dependent Probabilities
JEL Classification: C61, C63, O40

In memory of Tapan Mitra, with whom Fabio Privileggi coauthored some early pioneering works on stochastic one-sector optimal growth models and the possible appearance of Cantor-like attractors supporting singular invariant probability distributions.

1 Introduction

Over the last decades, following the seminal work by Brock and Mirman (1972), a large and growing number of works has tried to characterize the implications of stochasticity on macroeconomic dynamics and economic growth (see Olson and Roy, 2005, for a survey). Several of these studies analyze the eventual fractal nature of the steady state in traditional macroeconomic frameworks, which are now well known to give rise to random dynamics possibly converging to invariant measures supported on fractal sets (Montrucchio and Privileggi, 1999). Indeed, in a classical discrete time one-sector Ramsey (1928) model with logarithmic utility and Cobb-Douglas production in which output is affected by binary random shocks, the optimal economic dynamics can be converted into affine iterated function systems converging to invariant probability measures, which may turn out to be either singular and supported on a Cantor-like set or absolutely continuous (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004; 2006; 2009; La Torre et al., 2015). Several extensions of the standard setup have been developed over the years in order to consider multi-sector frameworks, to allow for sustained endogenous growth, to permit shocks to affect factor shares, and to account for pollution externalities, showing that even in such contexts similar results apply apart from the fact that the support of the invariant probability measure may be some other fractal set, like the Sierpinski gasket or the Barnsley's fern (La Torre et al., 2011, 2015, 2018b, 2018c).

To the best of our knowledge, all the refinements and extensions of the classical stochastic optimal growth model rely upon the assumption that the probability with which shocks occur is constant. Even if this setting is useful to characterize macroeconomic dynamics in a simple and intuitive way, it limits the analysis of the implications of important issues, such as the economic inefficiency induced by uncertainty. Indeed, whenever the shocks probability is constant the centralized and decentralized solutions perfectly coincide and uncertainty does not generate any efficiency loss. Whenever the shocks probability is not constant but endogenously changes with economic conditions, individual agents may fail to internalize the way their decisions affect the probability of shocks realization which in turn drives capital accumulation, resulting eventually in the introduction of some important inefficiency. The presence of such a distortion between the centralized and decentralized solutions explains why in reality policymakers do play an essential role in favoring the achievement of the first-best. Understanding thus how reconciling real world observations with macroeconomic theory is crucial to develop a realistic theory of economic growth and development. This paper wishes to make a first contribution in this direction by extending the classical optimal stochastic growth model to allow probabilities to be state-dependent, that is to depend on the level of the capital stock. State-dependent probabilities are a natural generalization of constant probabilities which allow to explain the mechanisms through which uncertainty might be a source of efficiency loss.

Specifically, we extend the classical discrete time stochastic one-sector optimal growth model with logarithmic utility and Cobb-Douglas production á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. We assume that the probability of occurrence of different shocks depends on the capital stock, and thus as the economy accumulates capital the probability of realization of given stocks endogenously changes. We consider the state-dependent probability to be a monotonic function of capital, analyzing how results may change in situations in which the probability increases or decreases with capital. By assuming that the probability function takes a logarithmic form, we are able to explicitly characterize the optimal solution of such an extended optimal growth model, discussing how the centralized and decentralized solutions differ. We show that since in a centralized setting state-dependent probabilities affect the optimal policy, they act as an engine of capital accumulation, which through its effects on the probability of shocks realization crucially drives the steady state capital distribution. However, since in a decentralized setting such effects are not accounted for, the optimal policy turns out be independent of state-dependent probabilities, introducing a wedge between the centralized and decentralized outcomes. In particular, whenever the probability is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, the decentralized economy underinvests (overinvests) in capital accumulation compared to the first-best, leading the steady state capital distribution to be characterized by a leftward (rightward) shifted support. This result generalizes those traditionally discussed in the stochastic optimal growth literature (Brock and Mirman, 1972; Montrucchio and Privileggi, 1999; Mitra et al., 2003), as the wedge between the centralized and decentralized solutions vanishes whenever probabilities do not depend on the capital stock. We also show how the features of state-dependent probabilities affect the spread and shape of such a steady state distribution, which tends to be more skewed (more evenly spread) whenever the probability decreases (increases) with capital. We also show that the optimal dynamics can be converted into a contractive affine iterated function system (IFS) with affine state-dependent probabilities (SDP) which, under rather general conditions, converges to an invariant self-similar measure supported on a (possibly fractal) compact attractor. This result generalizes those presented in the fractal steady state and stochastic optimal growth literature (Montrucchio and Privileggi, 1999; Mitra et al., 2003; La Torre et al., 2015), which has shown that under constant probabilities the optimal dynamics can be transformed in a traditional IFS in which probabilities are not state-dependent.

Despite the fact that the probability of shocks realization may depend on the level of some state variable is a very intuitive and natural framework to consider, the role of state-dependent probabilities has not been explored in depth thus far. State-dependent probabilities and in particular IFSSDP have received much attention in the mathematics literature (Barnsley et al., 1988; Stenflo, 2002), but they have only seldom been discussed in economics (La Torre et al., 2019, 2023). La Torre et al. (2019) discuss the implications of state-dependent probabilities on the possible steady state outcome in a purely dynamic economic growth model with health capital (abstracting completely from optimizing behavior) in which the probability of shocks depends on the relative abundance of health capital with respect to physical capital. La Torre et al. (2023) instead introduce optimizing behavior in an economic-epidemiological framework in which random shocks associated with the diffusion of a new disease strain occur with probabilities depending on the level of disease prevalence. They analyze how the steady state distribution of disease prevalence is affected by the characteristics of the state-dependent probability function and by optimal policymaking in a first-best scenario. Different from them we consider a macroeconomic framework by focusing on an optimal growth setup in which we analyze how the first- and second-best outcomes compare to clarify the implications of state-dependent probabilities on the gap in the steady state capital distribution between the centralized and decentralized scenarios.

The paper proceeds as follows. Section 2 introduces our extended Brock and Mirman's (1972) model with state-dependent probabilities, distinguishing between situations in which the probabilities are either decreasing or increasing with the capital stock. Section 3 derives the optimal solution of the social planner's problem discussing how the optimal policy changes (with respect to the standard one under constant probabilities) because of the presence of state-dependent probabilities. Section 4 determines the solution of the decentralized problem comparing it with the first-best outcome to highlight the inefficiency introduced by state-dependent probabilities. Section 5 characterizes some properties of the steady state distribution in relation to convergence and singularity vs. absolute continuity, presenting some robustness checks of our main conclusions. Section 6 as usual presents concluding remarks and highlights directions for future research. A brief review of the theory of IFS is presented in Appendix A, while all the proofs of our main results are reported in Appendix B.

2 The Model

We extend the classical discrete time stochastic one-sector growth model á-la Brock and Mirman (1972), with logarithmic utility and Cobb-Douglas production function, to allow probabilities to be state-dependent. The optimization problem can be summarized by the following stochastic dynamic programming model:

$$\begin{aligned} V(k_0, z_0) &= \max_{c_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \\ &\text{s.t. } k_{t+1} = z_t k_t^{\alpha} - c_t \\ &k_0 > 0 \text{ and } z_0 \in \{r, 1\} \text{ given,} \end{aligned} \tag{1}$$

where $\mathbb{E}_0$ is the expectation operator at time $t = 0$, k_t capital, c_t consumption, $0 < \alpha < 1$ the capital share, $0 < \beta < 1$ the discount factor, and $\{z_t\}_{t=0}^{\infty}$ a Bernoulli process taking values $0 < r < 1$ and 1 with probabilities $p(k_t)$ and $1 - p(k_t)$, respectively. Therefore, at each time t, the productivity shock z_t can take only two values with state-dependent probabilities, and in particular the fact that probabilities depend on the capital level implies that the realization of shocks is related to the past evolution of capital, suggesting that economic development may be characterized by either monotonic or non-monotonic dynamics. Specifically, whenever $p' < 0$ at low (high) capital levels the probability of the worst shock realization is high (low) and this tends to prevent (boost) capital accumulation deterring (inducing) economic growth. Overall this may give rise to a persistence of negative (positive) growth periods characterizing monotonic economic dynamics. Conversely, whenever $p' > 0$, at low capital levels the probability of the worst shock realization is low and this tends to favor capital accumulation promoting economic growth, but as capital increases so does the probability of the worst shock realization which tends to reduce capital accumulation resulting eventually in negative growth. Overall this may give rise to an alternate sequence of positive and negative growth periods characterizing non-monotonic economic dynamics. These two alternative situations represent diametrically different macroeconomic scenarios. The $p' < 0$ case characterizes a framework in which productivity shocks are procyclical, while the $p' > 0$ case a setting in which productivity shocks are countercyclical. Several studies discuss why the cyclicality of shocks may change over time or vary across countries, and this may be related to changes in factor utilization, the flexibility of labor and capital markets, the structural changes associated with economic development, and the effectiveness of macroeconomic stabilization policies (Fernald and Wang, 2016; Mayer et al., 2018). Note that whenever $p' = 0$ probabilities are constant, restoring the classical Brock and Mirman's (1972) model in which such alternative outcomes are not possible, so that productivity shocks are completely acyclical and capital accumulation resembles a random walk. In order to understand how different characteristics of the state-dependent probabilities might affect macroeconomic dynamics, in the remainder of the paper we will focus on a setup in which the relation between p and k_t is monotonic analyzing how the results may change when either $p' \leq 0$ or $p' \geq 0$.

The reduced problem associated with (1) can be stated as follows:

$$V(k_0, z_0) = \max_{k_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln(z_t k_t^\alpha - k_{t+1}) \tag{2}$$
$$\text{s.t. } 0 \le k_{t+1} \le z_t k_t^\alpha$$
$$k_0 > 0 \text{ and } z_0 \in \{r, 1\} \text{ are given.}$$

Note that the probability $p(k_t)$ determines the occurrence of the random shock z_t at the same time t in which the actual amount of capital k_t is employed in production; in this scenario production occurs after the shock z_t is realized, and its occurrence is controlled by the state-dependent probability $p(k_t)$ depending on the actual availability of the stock of capital k_t in the same period t. However, as the amount of capital available at time t corresponds to the investment decision made at time $t-1$, such an assumption actually determines in essence a Markov-type stochastic dynamic for capital, in which the probability of the random variable z_t at time t depends on a choice made in the previous period $t-1$.

It is straightforward to verify that (2) is a *concave problem* as the z_t-sections of the graph $G = \{(k_t, k_{t+1}, z_t) : k_{t+1} \in \Gamma(k_t, z_t)\}$ of the optimal correspondence $\Gamma(k_t, z_t) = \{k_{t+1} : 0 \le k_{t+1} \le z_t k_t^\alpha\}$ are convex sets. Moreover, the dynamic constraint $\Gamma(k_t, z_t)$ eventually (monotonically) leads any feasible trajectory $\{k_t\}_{t=1}^{\infty}$ inside the interval $[0, 1]$ as time elapses, because $z_t k_t^\alpha \le k_t^\alpha < k_t$ for any value $k_t > 1$. That is, the trapping region for the dynamics that are admissible for problem (2) is the interval $[0, 1]$, so that, without loss of generality, by assuming that the initial capital value k_0 lies in such an interval, any trajectory will have values that remain confined in it.

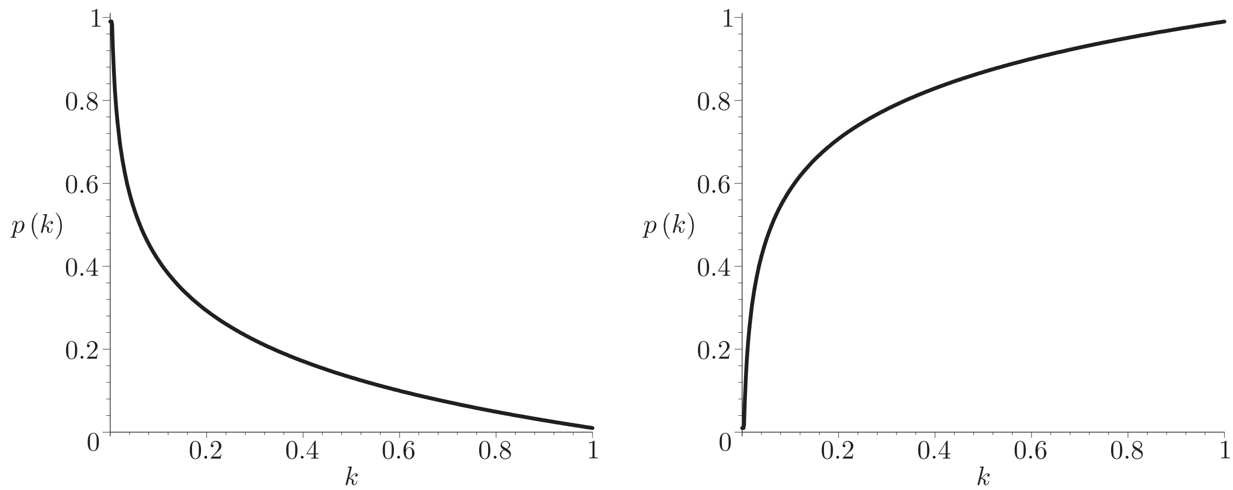
In order to explicitly solve the optimization problem above we need to specify the functional form of the state-dependent probability function. As a matter of analytical tractability, we assume a logarithmic form $p(k) = A + B \ln k$ for the state-dependent probability. Of course, any such logarithmic forms turn out to be unbounded over the interval $(0, 1]$, while state-dependent probabilities must satisfy $0 < p(k) < 1$ for any feasible state value k. We overcome such an issue by opting for a piecewise functional form that is constant for k values close to 0 while taking the form $p(k) = A + B \ln k$ for larger k values, so to keep the probability bounded between 0 and 1. Recall that, by assuming that the initial capital value k_0 lies in the interval $[0, 1]$, any trajectory will have values that remain confined in it. Under such an assumption we can introduce the following two piecewise-logarithmic forms for the state-dependent probability, one decreasing and one increasing in k, defined for $k \in [0, 1]$:

$$p(k) = \begin{cases} 1-\delta & \text{if } 0 \le k < e^{-\frac{1-\delta-\gamma}{\varepsilon}} \\ \gamma - \varepsilon \ln k & \text{if } e^{-\frac{1-\delta-\gamma}{\varepsilon}} \le k \le 1 \end{cases} \tag{3}$$

$$p(k) = \begin{cases} \delta & \text{if } 0 \le k < e^{-\frac{1-\delta-\gamma}{\varepsilon}} \\ 1-\gamma + \varepsilon \ln k & \text{if } e^{-\frac{1-\delta-\gamma}{\varepsilon}} \le k \le 1, \end{cases} \tag{4}$$

with $\delta, \gamma > 0$ such that $\delta + \gamma < 1$ and $\varepsilon > 0$ sufficiently small.

Clearly, as $k \le 1$, (3) defines a (Lipschitz) continuous state-dependent probability which satisfies $0 < p(k) < 1$ for all $0 \le k \le 1$, is constant over $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ and strictly decreasing in k over $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, while (4) defines a continuous state-dependent probability which again satisfies $0 < p(k) < 1$ for all $0 \le k \le 1$, is constant over $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ and strictly increasing in k over $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. Note that, for any fixed $\delta, \gamma > 0$ satisfying $\delta + \gamma < 1$, ε can be chosen small enough so to have the (more relevant) interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}},1\right]$ arbitrarily large; we shall return on this property later on. Figure 1 shows an example of the probability functions according to (3) and (4) for $\delta=\gamma=0.01$ and $\varepsilon=0.1756$; for such parameters' values the kink point turns out to be $e^{-\frac{1-\delta-\gamma}{\varepsilon}}=0.0038$.



(a) (b)

FIGURE 1: state-dependent probabilities for $\delta=\gamma=0.01$ and $\varepsilon=0.1756$; a) as defined in (3), b) as defined in (4).

In the next two sections we will characterize the optimal solution of problem (2) both in centralized and decentralized settings to understand whether the state-dependency of probabilities may generate some inefficiency and eventually to assess how the nature of such inefficiency may change with the features of the state-dependent probability function.

3 The Centralized Setting

We start by analyzing the centralized setting in which the social planner effectively accounts for the state-dependency of probabilities. Indeed, a social planner understands that their capital investment affects the probability of shocks realization, thus they internalize the effects of capital accumulation on future capital values induced by the state-dependent features of the probability function. Therefore, in the centralized framework the Bellman equation associated to (2) reads as:

$$V\left(k,z\right)=\max_{0\leq y\leq zk^{\alpha}}\left[\ln\left(zk^{\alpha}-y\right)+\beta\mathbb{E}_{y}V\left(y,z'\right)\right],$$

where $\mathbb{E}_y$ denotes the expectation operator that depends on the probabilities of both realizations of the random variable z' occurring in the next period, itself depending on the saving choice y, which corresponds to the capital available in the next period, that is, $\Pr\left(z'=r\right)=p\left(y\right)$, while $\Pr\left(z'=1\right)=1-p\left(y\right)$—recall that, for given y, the random variable z' is independent of past realizations. Then, the expectation $\mathbb{E}_y$ can be directly evaluated and the above equation can be rewritten in the following form:

$$V\left(k,z\right)=\max_{0\leq y\leq zk^{\alpha}}\left\{\ln\left(zk^{\alpha}-y\right)+\beta p\left(y\right)V\left(y,r\right)+\beta\left[1-p\left(y\right)\right]V\left(y,1\right)\right\}. \tag{5}$$

We search for a closed-form solution for the Bellman equation by applying the "Guess and Verify" Method (Stokey and Lucas, 1989; Bethmann, 2007; La Torre et al., 2015) in order to determine a closed-form expression for the value function in (5) in situations in which $p(k_t)$ is defined as either in (3) or in (4). Following previous literature we guess the following form for the value function in (5):

$$V(k,z) = A + B\ln k + C\ln z,$$

where A, B and C are constants to be determined. For such a logarithmic guess the Bellman equation in (5) becomes:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y\le zk^{\alpha}} \left[\ln(zk^{\alpha}-y) + \beta(A + B\ln y) + \beta p(y) C\ln r\right]. \end{aligned} \tag{6}$$

Both state-dependent probabilities $p(y)$ with the forms defined either in (3) or in (4) are not differentiable at $y = e^{-\frac{1-\delta-\gamma}{\varepsilon}}$; hence, provided that $zk^{\alpha} \ge rk^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, in both cases we must consider two different Bellman equations of the type in (6) depending on whether $y \in \left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ or $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, zk^{\alpha}\right]$. Specifically, when $p(y)$ is defined according to (3), the above equation becomes:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y< e^{-\frac{1-\delta-\gamma}{\varepsilon}}} \left[\ln(zk^{\alpha}-y) + \beta A + \beta B\ln y + \beta(1-\delta) C\ln r\right], \end{aligned} \tag{7}$$

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}}\le y\le zk^{\alpha}} \left[\ln(zk^{\alpha}-y) + \beta A + \beta B\ln y + \beta(\gamma - \varepsilon\ln y) C\ln r\right], \end{aligned} \tag{8}$$

while when $p(y)$ is defined according to (4), it takes the form:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y< e^{-\frac{1-\delta-\gamma}{\varepsilon}}} \left[\ln(zk^{\alpha}-y) + \beta A + \beta B\ln y + \beta\delta C\ln r\right], \end{aligned} \tag{9}$$

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}}\le y\le zk^{\alpha}} \left[\ln(zk^{\alpha}-y) + \beta A + \beta B\ln y + \beta(1-\gamma + \varepsilon\ln y) C\ln r\right]. \end{aligned} \tag{10}$$

Equations (7) and (9) represent problems that keep probabilities constant forever ($p = 1-\delta$ and $1-p = \delta$ in the former, $p = \delta$ and $1-p = 1-\delta$ in the latter); however, if, after a finite number of iterations, k_t becomes larger than $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, the relevant Bellman equations become those defined in (8) and (10). Therefore, equations (7) and (9) turn out to be completely useless unless we can guarantee that $k_t < e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ forever, that is, for every $t \ge 0$. Because $\delta + \gamma < 1$, for ε sufficiently small the term $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ can be made arbitrarily small, which, in turn, implies that the possibility of k_t jumping above the level $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ after a finite number of iterations becomes likely. As a matter of fact, the Inada conditions exhibited by the lower Cobb-Douglas production function, rk_t^{α}, invites the social planner to choose investment levels k_{t+1} much larger than the actual stock of capital k_t available at time t when the latter is very close to the left-end point 0 of the feasible set $[0,1]$, thus easily leading to a value $k_{t+1} > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$.

For k values in $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ problem (2) turns out to be a standard stochastic intertemporal model with constant probabilities, either $p = 1-\delta$ and $1-p = \delta$ or $p = \delta$ and $1-p = 1-\delta$. Hence, in this scenario we can invoke the well known result for this class of problems and easily find that, whenever $k \in \left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ the optimal policy yields the optimal investment given by (see, e.g., Mitra et al., 2003; Stokey and Lucas, 1989):

$$y^* = h(k, z) = \alpha\beta z k^{\alpha}. \tag{11}$$

Now, if ε is chosen sufficiently small with respect to parameters α, β and r, after a finite number τ of iterations of the policy (11) the optimal short-run trajectory will reach a value $k_\tau > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. The next assumption identifies such a threshold value for ε.

A. 1 *Parameters $\delta, \gamma, \varepsilon$ satisfy $\delta, \gamma, \varepsilon > 0$ and $\delta + \gamma < 1$. Moreover ε is small enough to satisfy:*

$$\varepsilon < -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)}. \tag{12}$$

Note that the RHS in (12) is positive as $1-\delta-\gamma > 0$ and $\ln(\alpha\beta r) < 0$.

Lemma 1 *Under Assumption A.1—specifically, condition (12)—the regime represented by both Bellman equations in (7) and (9) cannot be sustained over time, as there exist a finite number of iterations $\tau \geq 0$ such that the optimal capital value in that iteration satisfies $k_\tau \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$.*

In view of Lemma 1, in the following we shall assume that Assumption A.1 holds and that the initial capital stock satisfies $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, and focus exclusively on the (truly) state-dependent case represented by the second-type Bellman equations (8) and (10) over the (compact) interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. The next Propositions 1 and 2 will establish that, under such assumptions, the optimal capital trajectory k_t^* remains confined in the interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$ for all $t \geq 0$ indeed, thus justifying the focus exclusively on the relevant Bellman equations (8) and (10).

We consider first the case characterized by the decreasing state-dependent probability defined in (3) for $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$: $p(y) = \gamma - \varepsilon \ln y$. In this case the (relevant) Bellman equation (8) reads as:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}} \leq y \leq zk^{\alpha}} \left[\ln(zk^{\alpha} - y) + \beta(A + \gamma C \ln r) + \beta(B - \varepsilon C \ln r)\ln y\right]. \end{aligned} \tag{13}$$

It is then possible to prove the following result.

Proposition 1 *Under Assumption A.1 and for $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, the solution of the Bellman equation (13) is the function:*

$$V(k,z) = A + B\ln k + C\ln z$$

where:

$$A = \frac{\ln[1-\beta(\alpha-\varepsilon\ln r)]}{1-\beta} + \frac{\beta(\alpha-\varepsilon\ln r)\ln[\beta(\alpha-\varepsilon\ln r)] + \beta\gamma\ln r}{(1-\beta)[1-\beta(\alpha-\varepsilon\ln r)]}, \tag{14}$$

$$B = \frac{\alpha}{1-\beta(\alpha-\varepsilon\ln r)}, \tag{15}$$

$$C = \frac{1}{1-\beta(\alpha-\varepsilon\ln r)}; \tag{16}$$

the optimal policy for capital is given by:

$$k_{t+1}^* = h(k_t^*, z_t) = \beta(\alpha - \varepsilon \ln r) z_t (k_t^*)^\alpha, \tag{17}$$

while the corresponding optimal policy for consumption is given by:

$$c_t^* = [1 - \beta(\alpha - \varepsilon \ln r)] z_t (k_t^*)^\alpha. \tag{18}$$

It is possible to show (see Appendix B) that $\beta(\alpha - \varepsilon \ln r) < 1$, which ensures that: i) coefficients B in (15) and C in (16) are strictly positive, which, in turn, imply that the value function $V(k, z)$ solving equation (13) is strictly concave in k and that the RHS is strictly concave in y, so that the optimal policy in (17) is unique; and ii) the optimal consumption in (18) is strictly positive. Therefore, Proposition 1 determines the unique optimal policy associated with our extended Brock and Mirman's (1972) model with decreasing state-dependent probabilities. We can note that the optimal policy in (17) differs from the standard (under constant probability) optimal policy $k_{t+1}^* = h(k, z) = \alpha\beta z k^\alpha$ as in (11) because of the role of the state-dependent probability $p(k) = \gamma - \varepsilon \ln k$ as in (3). Specifically, the positive term added to the original multiplicative coefficient α appearing in (11) (*i.e.*, $-\varepsilon \ln r$) takes into account that, as $p(k) = \gamma - \varepsilon \ln k$ is decreasing in k, investing more in future capital increases the probability $1 - p(k)$ of having future favorable shocks $z_t = 1$. Clearly, if $\varepsilon = 0$, that is, the probability $p(k)$ does no longer depend on capital, the optimal policy (17) perfectly coincides with the standard one in (11).

We now move to the case of an increasing state-dependent probability defined in (4) for $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$: $p(y) = 1 - \gamma + \varepsilon \ln y$. In this case the (relevant) Bellman equation (10) reads as:

$$\begin{aligned} V(k, z) &= A + B \ln k + C \ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}} \le y \le z k^\alpha} \left[\ln(z k^\alpha - y) + \beta[A + (1-\gamma) C \ln r] + \beta(B + \varepsilon C \ln r) \ln y\right]. \end{aligned} \tag{19}$$

Unlike the case with decreasing state-dependent probabilities, now we need an additional condition on parameter ε in the definition of probability in (4) – the following condition (20) – that guarantees interiority of the optimal policy (24) determined in the next Proposition 2 whenever its argument $k_t^* \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. In fact, now the term $\varepsilon \ln r < 0$ indicates that, when the state-dependent probability is increasing, the optimal choice on investment turns out to be strictly lower than that prescribed by the standard optimal policy (11). This property requires that the upper bound for parameter ε in condition (12) is further restricted in order to assure that the optimal trajectory generated by (24) remains trapped in the (open) interval $\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right)$ for all $t \ge 0$.

A. 2 *Under Assumption A.1, suppose that ε is sufficiently small to satisfy:*

$$e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\varepsilon < \alpha\beta r. \tag{20}$$

Condition (20), although stated in implicit form with respect to ε, is meaningful, as the RHS is strictly positive and the LHS is strictly positive, strictly increasing in ε and approaches 0 as $\varepsilon \to 0^+$. In other words, for any choice for $0 < \alpha, \beta, \delta, \gamma, r < 1$ satisfying all our assumptions, there always exist some values $\varepsilon > 0$ satisfying (20). Its threshold upper bound value is the unique $\varepsilon > 0$ satisfying (20) with equality. Moreover, as $-(\beta r \ln r)\varepsilon > 0$, condition (20) is stricter than (*i.e.*, implies) condition (12); indeed, $e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\varepsilon < \alpha\beta r \Longrightarrow e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} < \alpha\beta r$, where the last inequality is equivalent to (12). Therefore, Lemma 1 always holds true under Assumption A.2.

Proposition 2 *Under Assumption A.2 and for* $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$*, the solution of the Bellman equation (19) is the function:*

$$V(k, z) = A + B \ln k + C \ln z$$

where:

$$A = \frac{\ln[1-\beta(\alpha+\varepsilon\ln r)]}{1-\beta} + \frac{\beta(\alpha+\varepsilon\ln r)\ln[\beta(\alpha+\varepsilon\ln r)] + \beta(1-\gamma)\ln r}{(1-\beta)[1-\beta(\alpha+\varepsilon\ln r)]}, \tag{21}$$

$$B = \frac{\alpha}{1-\beta(\alpha+\varepsilon\ln r)}, \tag{22}$$

$$C = \frac{1}{1-\beta(\alpha+\varepsilon\ln r)}; \tag{23}$$

the optimal policy for capital is given by:

$$k^*_{t+1} = h(k^*_t, z_t) = \beta(\alpha+\varepsilon\ln r) z_t (k^*_t)^\alpha, \tag{24}$$

while the corresponding optimal policy for consumption is given by:

$$c^*_t = [1-\beta(\alpha+\varepsilon\ln r)] z_t (k^*_t)^\alpha. \tag{25}$$

Also in this case it is possible to show (Appendix B) that under condition (20) $0 < \beta(\alpha+\varepsilon\ln r) < 1$ holds, thus assuring that: i) the optimal investment in (24) is strictly positive; ii) the optimal consumption in (25) is strictly positive; and iii) coefficients B in (22) and C in (23) are both strictly positive, which, in turn, together with the property $\beta(\alpha+\varepsilon\ln r) > 0$, imply that the value function $V(k, z)$ solving equation (19) is strictly concave in k and that the RHS is strictly concave in y, so that the optimal policy in (24) is unique. Therefore, Proposition 2 determines the unique optimal policy associated with our model with increasing state-dependent probabilities. We can note that also in the case of increasing probability the optimal policy in (24) differs from the standard (under constant probability) optimal policy $k^*_{t+1} = h(k, z) = \alpha\beta z k^\alpha$ in (11) because of the effects of the state-dependent probability $p(k) = 1-\gamma+\varepsilon\ln k$ as in (4). The negative term added to the original multiplicative coefficient α appearing in (11) (*i.e.*, $+\varepsilon\ln r$), emphasizes the fact that, as $p(k) = 1-\gamma+\varepsilon\ln k$ is increasing in k, the social planner takes into account that too large an investment increases the probability $p(k)$ of bad shocks $z_t = r$ occurring in subsequent times that will cause a reduction in the future capital stock. Also in this case, whenever $\varepsilon = 0$ the probability $p(k)$ turns out not to depend on capital any longer, and thus the optimal policy (24) perfectly coincides with the standard one in (11).

Comparing the optimal policies (17) and (24) under decreasing and increasing state-dependent probabilities respectively, it is straightforward to notice that they differ only for the additive term $\varepsilon\ln r$, whose sign is positive in the former case and negative in the latter case, such that the optimal policy prescribes a larger (smaller) investment whenever the probability decreases (increases) with the capital stock. However, independently of whether the probability increases or decreases with the capital stock, the optimal policy under state-dependent probability crucially depends on the shocks probability $p(k_t)$. Therefore, by affecting the optimal capital dynamics, state-dependent probabilities act as an engine of capital accumulation, which through its effects on the probability of shocks realization impacts the evolution of capital over time. Such effects are completely absent under the standard constant probability assumption, since in the standard Brock and Mirman's (1972) setup capital dynamic is completely independent of the (constant) probability.

Since affecting the time evolution of capital, state-dependent probabilities affect also its long run steady state distribution. Indeed, the optimal policies (17) and (24) derived in Propositions 1 and 2 can be rewritten as follows, respectively:

$$k_{t+1} = \begin{cases} \theta_1 r k_t^\alpha & \text{with probability } p_1(k_t) \\ \theta_1 k_t^\alpha & \text{with probability } 1 - p_1(k_t) \end{cases} \tag{26}$$

and:

$$k_{t+1} = \begin{cases} \theta_2 r k_t^\alpha & \text{with probability } p_2(k_t) \\ \theta_2 k_t^\alpha & \text{with probability } 1 - p_2(k_t) \end{cases}, \tag{27}$$

where $\theta_1 = \beta(\alpha - \varepsilon \ln r)$, $p_1(k_t) = \gamma - \varepsilon \ln k_t$ as in (3), $\theta_2 = \beta(\alpha + \varepsilon \ln r)$, and $p_2(k_t) = 1 - \gamma + \varepsilon \ln k_t$ as in (4). Equations (26) and (27) characterize two nonlinear IFSSDPs which, by relying on the IFS theory (see Appendix A for a brief review), converge to a unique invariant distribution $\bar{\mu}$ supported on an interval (possibly the whole interval) whose endpoints are the fixed points of the two nonlinear maps $w_l(k) = \theta_i r k^\alpha$ (lower map) and $w_h(k) = \theta_i k^\alpha$ (higher map) for $i = 1, 2$, given by $\left[(\theta_1 r)^{\frac{1}{1-\alpha}}, \theta_1^{\frac{1}{1-\alpha}}\right]$ and $\left[(\theta_2 r)^{\frac{1}{1-\alpha}}, \theta_2^{\frac{1}{1-\alpha}}\right]$, respectively. Note that, due to the Inada condition of the Cobb-Douglas production function, k^α, the derivative of the higher map $w_h(k) = \theta_i k^\alpha$, in each IFS evaluated at the left endpoint of their attractor may be larger than 1;[1] in such circumstances, all the associated nonlinear IFSSDPs turn out to be not contractive and, in principle, convergence to a unique invariant measure may not be guaranteed. However, convergence to a unique invariant distribution is established by the fact that they are topologically conjugate of an affine IFSSDP, which, being a contraction, converges to a unique invariant measure (see the IFSSDP (31) introduced in Section 5).

The expressions for the supports of the invariant measures associated with the optimal policies (17) and (24) clearly show the effects of the features of the state-dependent probabilities on the steady state distribution of capital. In particular, when the state-dependent probability is decreasing (increasing) the steady state distribution is spread over a larger (smaller) range of values characterized on average by higher (lower) capital.[2] This is consistent with the working mechanisms of state-dependent probabilities as an engine of capital accumulation: since the optimal policy requires larger (smaller) investment when the probability is decreasing (increasing), this results in the long run in higher (smaller) capital values on average. Apart from such effects on the support of the invariant measure, state-dependent probabilities may affect the shape of the steady state distribution, but unfortunately it is not possible characterizing explicitly how. Therefore, in the following we will rely on a numerical approach to shed some light on this issue.

Specifically, we numerically approximate the evolution of a given probability distribution over time according to our nonlinear IFSSDPs (26) and (27) associated to the optimal policies (17) and (24), which solve (2). To this purpose, we apply a Maple algorithm[3] that approximates successive iterations of the Markov operator given by (39) in Appendix A associated with the relevant IFSSDP based on Algorithm 1 in La Torre et al. (2019), in order to have a qualitative idea on what the invariant distribution $\bar{\mu}$ may look like. In our benchmark parametrization (see Section 5 for a robustness analysis of our numerical results) we set the following values:

$$\alpha = 0.5, \qquad \beta = 0.96, \qquad r = 0.25, \qquad \delta = \gamma = 0.01 \qquad \text{and} \qquad \varepsilon = 0.1756. \tag{28}$$

[1] To be precise, this occurs whenever $\alpha > r$, as in all our numerical simulations below.

[2] To see this, note that, as $0 < \alpha, r < 1$, $\theta_1 > \theta_2 \Longrightarrow \theta_1^{\frac{1}{1-\alpha}} - (\theta_1 r)^{\frac{1}{1-\alpha}} > \theta_2^{\frac{1}{1-\alpha}} - (\theta_2 r)^{\frac{1}{1-\alpha}}$.

[3] The detailed code is available upon request.

Note that the value $\varepsilon = 0.1756$ is 0.0001 less than its upper bound 0.1757 defined by condition (20). The maps in the IFSSDPs (26) and (27) turn out to be $k_{t+1} = \theta_1 z_t k_t^{\alpha} = (0.7137)\, z_t k_t^{0.5}$ and $k_{t+1} = \theta_2 z_t k_t^{\alpha} = (0.2463)\, z_t k_t^{0.5}$ for $z_t \in \{0.25, 1\}$ respectively, and their attractors turn out to be the interval $\left[(r\theta_1)^2, \theta_1^2\right] = [0.0318, 0.5094]$ and $\left[(r\theta_2)^2, \theta_2^2\right] = [0.0038, 0.0606]$ respectively. In both the the IFSSDPs (26) and (27) the images of the two maps $w_l(k) = r\theta_i k^{0.5}$ (lower map) and $w_h(k) = \theta_i k^{0.5}$ (higher map) almost do not overlap, having in common only one point: $w_l(\theta_1^2) = w_h\left[(r\theta_1)^2\right] = 0.12736$ for the IFSSDP (26) and $w_l(\theta_2^2) = w_h\left[(r\theta_2)^2\right] = 0.0152$ for the IFSSDP (27). Therefore, in both cases the invariant measure has a full interval as support: [0.0318, 0.5094] and [0.0038, 0.0606] respectively. The plots of the state-dependent probabilities $p(k)$ for these parameters' values, both for the decreasing and for the increasing probability, are those reported in Figure 1 for the relevant interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0038, 1]$.

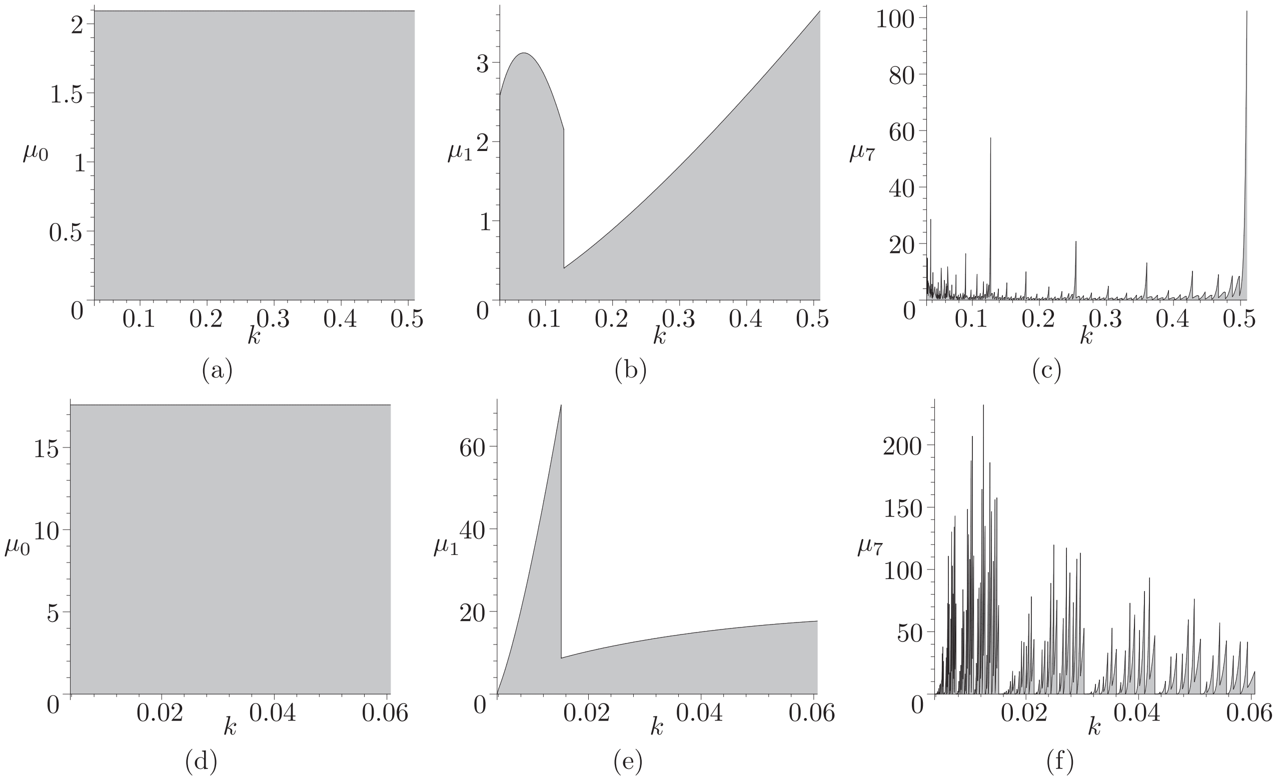


FIGURE 2: a) initial uniform density $\mu_0(k) \equiv 2.0938$ over [0.0318, 0.5094], b) 1^{st} and c) 7^{th} iterations of our Algorithm for the IFSSDP (26); d) initial uniform density $\mu_0(k) \equiv 17.59$ over [0.0038, 0.0606], e) 1^{st} and f) 7^{th} iterations of our Algorithm for the IFSSDP (27).

Figure 2 shows the initial uniform density $\mu_0(k) \equiv \frac{1}{\theta_i^2 - (r\theta_i)^2}$ (left panels), the 1^{st} (mid panels) and 7^{th} (right panels) iterations of our Maple algorithm for the IFSSDP (26) (top panels for $i = 1$, with $\mu_0(k) \equiv 2.0938$)—*i.e.* when the probability of the shock $z = r$ is decreasing and defined according to $p_1(k) = 0.01 - (0.1756)\ln k$—or for the IFSSDP (27) (bottom panels for $i = 2$, with $\mu_0(k) \equiv 17.59$)—*i.e.* when the probability of the shock $z = r$ is increasing and defined according to $p_2(k) = 0.99 + (0.1756)\ln k$. As convergence toward the unique invariant measure is geometric, *i.e.*, very fast, Figures 2(c) and 2(f), can be considered as good approximations of the invariant measures to which the IFSSDPs (26) and (27) converge asymptotically. From Figure 2(c) we learn that, as expected, $\bar{\mu}$ tends to concentrate most of the mass on k values close to the endpoints of $[0.0318, 0.5094]$. This is explained by the fact that a decreasing probability $p_1(k)$ introduces a conservative pattern for the k values, with a higher probability to either remain close to $k = 0.0318$ if the system is already there or to remain close to $k = 0.5094$ if the system is already in that area. This suggests that in the $p'(\cdot) < 0$ case economic development is characterized by a monotonic increase in the capital level which tends to be concentrated near one of (or both) the extremes of the support of the invariant measure, so that in steady state the outcome is associated with a high frequency of either large or small capital values, or both. Conversely, Figure 2(f) shows that an increasing probability like $p_2(k) = 0.99 + (0.1756)\ln k$ tends to concentrate more mass in the middle of the support $[0.0038, 0.0606]$, that is, future values of k are more likely to jump (almost) anywhere in the interval support than in the previous case. Again the justification of this pattern originates from the increasing probability $p_2(k)$ that raises the chance of the occurrence of the best shock $z = 1$ when k is small and viceversa. Therefore, in the $p'(\cdot) > 0$ case economic development is characterized by fluctuations in the capital level which tends to continually rise and fall leading on average to be dispersed but more densely concentrated toward the middle of the support of the invariant measure, so that the steady state outcome is associated with a state of diffuse capital levels.

We can clearly conclude that the property of increasingness or decreasingness of the state-dependent probabilities affects in a nontrivial way the steady state capital distribution. Indeed, it does not only determine the size of its support, which is wider and characterized by larger capital values in the $p' < 0$ case, but it impacts also on how the steady state distribution is spread over its support, which tends to be highly concentrated near one or both extremes of (more evenly spread over) the support in the $p' < 0$ ($p' > 0$) case. This also suggests that the optimal policy response to the procyclicality of productivity shocks tends to generate a more favorable long run outcome than the one we would observe in the case of countercyclical shocks.

4 The Decentralized Framework

We now move to the decentralized setting in which individual agents fail to account for the state-dependency of probabilities. Indeed, a single agent may not perceive that their individual capital investment affects the probability of shocks realization, thus they do not internalize the effects of capital accumulation on future capital values induced by the state-dependency of probabilities. Therefore, in the decentralized framework individual agents take the probability as given, forming thus their decision plans as if the probability of shocks realization were constant at the value $\bar{p}$. In such a framework, the Bellman Equation (5) associated to problem (2) when the shocks have constant probability $\bar{p}$, reads as follows:

$$V(k, z) = \max_{0 \leq y \leq zk^\alpha} \left[\ln(zk^\alpha - y) + \beta\bar{p}V(y, r) + \beta(1 - \bar{p})V(y, 1)\right],$$

It is well known (see, e.g., Mitra et al., 2003; Stokey and Lucas, 1989) that the optimal policy in this case is independent of the probability itself, as it is given by (11). The investment level given by this expression is the common choice of all individual agents in a decentralized economy, who fail to consider how the capital level affects the shock process through the state-dependent probabilities $p(k)$ defined as in (3) or (4). As it is different than both the social planner optimal policies (17) and (24), it characterizes a second-best, suboptimal solution for problem (2). By comparing the centralized and decentralized outcomes it is straightforward to conclude the following.

Proposition 3 *The centralized optimal policy prescribes a larger (smaller) capital investment than the decentralize policy whenever the state-dependent probability is decreasing (increasing).*

Proposition 3 states that the state-dependency of probabilities plays a crucial role in determining the extent of the difference between the centralized and decentralized solutions. If the state-dependent probability is decreasing in capital the decentralized economy underinvests in capital accumulation: by failing to internalize that capital accumulation reduces the probability of the worst shock realization individual agents devote less resources than the socially optimal level for the future, resulting thus eventually in a steady state distribution characterized by lower capital levels. If instead the state-dependent probability is increasing the decentralized economy overinvests in capital accumulation: by failing to internalize that capital accumulation increases the probability of the worst shock realization individual agents allocate more resources than the socially optimal level for the future, resulting thus eventually in a steady state distribution characterized by higher capital levels. Indeed, the second-best policy can be written in terms of a nonlinear IFSSDP as follows

$$k_{t+1} = \begin{cases} \alpha\beta r k_t^\alpha & \text{with probability } p_i(k_t) \\ \alpha\beta k_t^\alpha & \text{with probability } 1 - p_i(k_t) \end{cases} \quad \text{for } i = 1, 2, \tag{29}$$

with $p_1(k_t) = \gamma - \varepsilon \ln k_t$ as in (3) and $p_2(k_t) = 1 - \gamma + \varepsilon \ln k_t$ as in (4). Similarly to what we have discussed for the optimal policy, the IFSSDP (29) converges to an invariant measure whose support is a subset of the interval (possibly the whole interval) whose endpoints are the fixed points of the nonlinear maps $w_l(k) = \alpha\beta r k^\alpha$, $w_h(k) = \alpha\beta k^\alpha$, given by $\left[(\alpha\beta r)^{\frac{1}{1-\alpha}}, \alpha\beta^{\frac{1}{1-\alpha}}\right]$. It is straightforward to verify that such a support is leftward (rightward) shifted with respect to the support of the first-best distribution whenever the state-dependent probability is decreasing (increasing). In the classical Brock and Mirman (1972) framework, because the probability of shocks realization is constant, the centralized and decentralized solutions coincide, explaining why the standard macroeconomic theory—which abstracts from the state-dependency of probabilities—is not capable of justifying why in reality economic regulation is essential to restore the first-best outcome. As a matter of fact, if first- and second-best coincide, there is no need of regulation as competitive markets will achieve the first-best automatically.

As in the first-best framework, state-dependent probabilities affect also the shape of the steady state distribution; again, unfortunately, characterizing explicitly how is not possible and thus we need to rely on a numerical approach. By relying on the same parameter as in (28), the second-best policy turns out to be $k_{t+1} = \alpha\beta z_t k_t^\alpha = (0.48) z_t k_t^\alpha$, with associated IFSSDP (29) having the interval $\left[(\alpha\beta r)^2, (\alpha\beta)^2\right] = [0.0144, 0.2304]$ as attractor, all contained in the relevant interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0038, 1]$ for the state-dependent probabilities considered in our simulation and reported in Figure 1. Also the images of the two maps $w_l(k) = \alpha\beta r k^\alpha$ (lower map) and $w_h(k) = \alpha\beta k^\alpha$ (higher map) in the IFSSDPs (29) almost do not overlap, having in common only one point: $w_l\left[(\alpha\beta)^2\right] = w_h\left[(\alpha\beta r)^2\right] = 0.0576$. Therefore, the invariant measure has the full interval $[0.0144, 0.2304]$ as support. Note that, because $\theta_1 > \alpha\beta > \theta_2$, the attractor lies in a somewhat intermediate position between the attractors of the IFSSDPs (26) and (27).

Figure 3 shows the initial uniform density $\mu_0(k) \equiv \frac{1}{(\alpha\beta)^2 - (\alpha\beta r)^2} = 4.63$ (left panels), the 1^{st} (mid panels) and 7^{th} (right panels) iterations of our Maple algorithm for the IFSSDP (29) when the probability of the shock $z = r$ is decreasing and defined according to $p_1(k) = 0.01 - (0.1756) \ln k$ (top panels) or when the probability of the shock $z = r$ is increasing and defined according to $p_2(k) = 0.99 + (0.1756) \ln k$ (bottom panels).

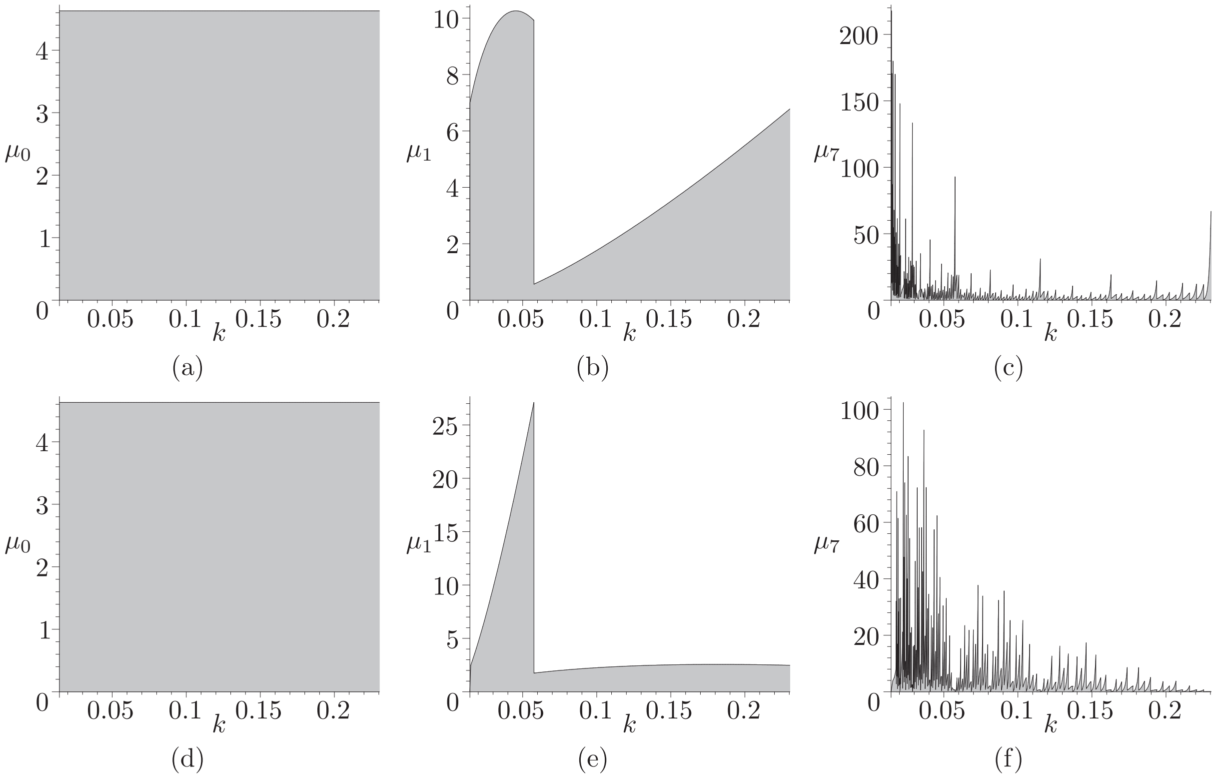


FIGURE 3: initial uniform density $\mu_0(k) \equiv 4.63$ over $[0.0144, 0.2304]$ (left), 1^{st} (mid) and 7^{th} (right) iterations of our Algorithm for the IFSSDP (29) whenever $p_1(k) = 0.01 - (0.1756)\ln k$ (top) or $p_2(k) = 0.99 + (0.1756)\ln k$ (bottom).

By comparing the approximations of the invariant measure generated by the IFSSDPs associated to the social planner first-best policies in Figures 2 with those associated to the decentralized second-best policies in Figure 3, it is interesting to observe that the latter exhibit a pattern which is consistent with the former. In fact, the characteristic pattern introduced by the Markov operator (39), clearly visible in Figures 2(b) and 2(e), is being replicated in Figures 3(b) and 3(e), as in both cases the same decreasing/increasing state-dependent probabilities are at work. Moreover, and more importantly, the high degree of concentration close to one of (or both) the endpoints of the support of the invariant measure exhibited by Figure 2(c), typical of the behavior generated by a decreasing state-dependent probability, is being maintained by Figure 3(c); similarly, both Figures 2(f) and 3(f) exhibit a fair degree of spreadness, as we should expect when the state-dependent probability is increasing.[4] However, a relatively higher spike on the left in Figure 3(b) than that in Figure 2(b) yields a dramatically different picture asymptotically: the approximation of the invariant measure in Figure 3(c) concentrates most of its weight close to the left endpoint of the support, instead of concentrating it close to the right endpoint, as occurs in Figure 2(c). Similarly, but less dramatically, a relatively lower spike on the left in Figure 3(e) than that in Figure 2(e) has the effect of slightly concentrating slightly more mass close to the left endpoint of the support of the approximation of the invariant measure in Figure 3(f) than what occurs in Figure 2(f).

[4]Note that the values on the vertical axis in Figures 2 and 3 should be considered in relative terms. Specifically, differences in spike values on the vertical axis in both figures are consistent with differences in diameters of the supports. For example, a spike of around 100 in Figure 2(c), where the diameter of the support is around 0.5, roughly corresponds to a spike higher than 200 in Figure 3(c), where the diameter is around 0.22.

Consistent with the fact that both Figures 3(c) and 3(f) represent invariant measures generated by suboptimal second-best dynamics, it is clear that in both scenarios considered for the decreasing and the increasing state-dependent probabilities such invariant measures concentrate more mass toward the lower endpoint of their support than their counterparts in Figures 2(c) and 2(f) do, thus determining a higher frequency of lower levels of capital in the long run due to suboptimal choice by the decentralized agents. Despite the difference in the support size and range of values, by normalizing the supports over the unit interval through a monotonically increasing transformation it is possible to conclude that decentralization leads to a stochastic steady state characterized on average by lower capital levels, independent of whether the state-dependent probabilities are either increasing or decreasing (see Section 5). In fact, the second-best policy $k_{t+1} = \alpha\beta z_t k_t^\alpha$ represents systematic underinvestment (overinvestment) when $p(k)$ is decreasing (increasing) that is when capital accumulation favors (deters) economic development, leading the economy to a worse outcome than that generated by the social planner first-best policy $k_{t+1} = \theta_i z_t k_t^\alpha$ for $i = 1, 2$.

5 Properties of the Invariant Measures

From our analysis thus far we have conjectured that the nonlinear IFSSDPs, (26) and (27) in the centralized framework and (29) in the decentralized context, have a steady state characterized by an invariant distribution supported on some closed set, but we have not proved anything yet in relation to its existence and the their eventual convergence to it. We now formally prove both claims by performing a variable change which recasts the above IFSSDPs in terms of a topologically equivalent linear IFSSDPs. Such a transformation allows us also to assess the singularity vs. absolute continuity properties of the invariant measure and to perform some robustness checks of our conclusions in relation to the features of the steady state distribution.

Recall that $\theta_1 = \beta(\alpha - \varepsilon \ln r)$, $\theta_2 = \beta(\alpha + \varepsilon \ln r)$; moreover, set $\theta_3 = \alpha\beta$. Consistent with extant literature (Montrucchio and Privileggi, 1999; Mitra et al., 2003; La Torre et al., 2019), it is straightforward to show that the following log-linear transformation:

$$x_t = -\frac{1-\alpha}{\ln r} \ln k_t + 1 + \frac{\ln \theta_i}{\ln r}, \qquad \text{for } i = 1, 2, 3, \tag{30}$$

defines an affine dynamic in the new variable x_t, which is topologically conjugate to the nonlinear map $k_{t+1} = \theta_i z_t k_t^\alpha$ in the capital variable k_t and has the interval $[0, 1]$ as trapping region for the dynamics associated to the first-best centralized solutions (17), (24) when the state-dependent probability is decreasing ($i = 1$) or increasing ($i = 2$), and to the (unique) second-best decentralized solution (11) with state-dependent probability either decreasing or increasing ($i = 3$). As $0 < r < 1$ and, by Propositions 1 and 2, $0 < \theta_i < 1$ for $i = 1, 2, 3$ as well, (30) are increasing affine transformations of $\ln k_t$ for $i = 1, 2, 3$. Specifically, they are continuous, invertible and each of them establishes a one-to-one correspondence between the nonlinear dynamics of k_t defined by the three maps (17), (24), (11) and the affine dynamics of the new variable x_t according to:

$$x_{t+1} = \alpha x_t + (1-\alpha)\left(1 - \frac{\ln z_t}{\ln r}\right),$$

which, in turn, can be rewritten in terms of the following IFSSDP:

$$x_{t+1} = \begin{cases} \alpha x_t & \text{with probability } p_j(x_t) \\ \alpha x_t + (1-\alpha) & \text{with probability } 1 - p_j(x_t), \end{cases} \tag{31}$$

where the *conjugate state-dependent probabilities* $p_j(x):[0,1]\to[0,1]$ are the affine functions defined in the next proposition.

Proposition 4 *For the probabilities defined in (3) and (4), under Assumptions A.1 and A.2 the conjugate state-dependent probabilities* $p_j(x):[0,1]\to[0,1]$ *associated to the IFSSDP (31) for* $j = FD$ *(first-best, decreasing),* FI *(first-best, increasing),* SD *(second-best, decreasing),* SI *(second-best, increasing) are:*

$$p_{FD}(x) = \gamma - \frac{\varepsilon}{1-\alpha}\ln(\theta_1 r) + \frac{\varepsilon\ln r}{1-\alpha}x \tag{32}$$

$$p_{FI}(x) = 1-\gamma + \frac{\varepsilon}{1-\alpha}\ln(\theta_2 r) - \frac{\varepsilon\ln r}{1-\alpha}x \tag{33}$$

$$p_{SD}(x) = \gamma - \frac{\varepsilon}{1-\alpha}\ln(\theta_3 r) + \frac{\varepsilon\ln r}{1-\alpha}x \tag{34}$$

$$p_{SI}(x) = 1-\gamma + \frac{\varepsilon}{1-\alpha}\ln(\theta_3 r) - \frac{\varepsilon\ln r}{1-\alpha}x. \tag{35}$$

All satisfy $0 < p_j(x) < 1$ *for all* $x \in [0,1]$*,* $p_{FD}(x)$ *and* $p_{SD}(x)$ *are strictly decreasing while* $p_{FI}(x)$ *and* $p_{SI}(x)$ *are strictly increasing.*

Different from what happens under constant probabilities, Proposition 4 states that with state-dependent probabilities also the probability function needs to be converted in an affine function in order to derive a topologically equivalent transformation of the original dynamical system. Specifically, $p_{FD}(x)$ is the affine state-dependent probability associated to the affine IFSSDP (31) when solution is given by the first-best map $k_{t+1} = \theta_1 z_t k_t^\alpha$ and the original state-dependent probability is decreasing; $p_{FI}(x)$ is the affine state-dependent probability associated to the affine IFSSDP (31) when the first-best solution is given by the first-best map $k_{t+1} = \theta_2 z_t k_t^\alpha$ and the original state-dependent probability is increasing; while $p_{SD}(x)$ and $p_{SI}(x)$ are the affine state-dependent probabilities associated to the affine IFSSDP (31) when the solution is given by the (unique) second-best map $k_{t+1} = \theta_3 z_t k_t^\alpha = \alpha\beta z_t k_t^\alpha$ and the original state-dependent probability is either decreasing or increasing. Note that Proposition 4 establishes that the IFS defined by the unique affine map $x_{t+1} = \alpha x_t + (1-\alpha)\left(1 - \frac{\ln z_t}{\ln r}\right)$ is topologically equivalent to all four dynamics defined by the nonlinear maps $k_{t+1} = \theta_i z_t k_t^\alpha$, including first- and second-best dynamics; the difference among all four is crucially and exclusively determined by which of the four probabilities (32)–(35) is being associated to it.

The IFSSDP (31) with associated state-dependent probabilities (32)–(35) can be analyzed through the tools from the IFS theory (see Appendix A), which ensure the existence and convergence of a unique stationary distribution $\bar{\mu}$ for such an IFSSDP. Since the unique IFS in (31) is topologically equivalent to either (26), (27) or (29) when the appropriate affine probabilities $p_j(x)$ are used, this confirms that the nonlinear IFSSDPs analyzed in the previous sections do converge to a unique stationary distribution as well. Note that the log-linear transformation (30), as it applies to both the first- and second-best policies, generates a normalization of the support of all their steady state distributions over the unit interval, allowing for a more straightforward comparison of the effects of different properties of the state-dependent probabilities on the characteristics of the invariant measure.

Consistent with the IFS theory, the linearity property of the IFSSDP (31) allows us also analyze the features of the invariant measure in terms of singularity or absolute continuity. Absolutely continuous measures can be represented by a density and thus admit a full representation depending only on a few parameters, while singular invariant measures do not have a simple and effective representation unless we state their value on every point of their domain (Mitra et al., 2003; La Torre et al., 2023). As absolutely continuous measures are more well-behaved than singular measures as they allow for a more precise forecasting of future dynamics, it may be useful characterize the conditions under which the invariant measure may be absolutely continuous. Theorem 5 in Appendix A can be directly applied to our IFSSDP (31) by setting $\beta = \alpha$, $\tau_1 = 0$ and $\tau_2 = 1 - \alpha$, and this allows us to conclude that the invariant distribution may be either singular if the capital share α is small (*i.e.*, $\alpha \leq 1/2$) or absolutely continuous if it is large (*i.e.*, $\alpha > 1/2$), and in particular absolute continuity requires that $\alpha > e^{\Theta}$, namely that the capital share exceeds a certain value, where:

$$\Theta = \max\left\{p_{\sup} \ln\left(p_{\sup}\right), p_{\inf} \ln\left(p_{\inf}\right)\right\} + \\ + \max\left\{\left(1 - p_{\sup}\right) \ln\left(1 - p_{\sup}\right), \left(1 - p_{\inf}\right) \ln\left(1 - p_{\inf}\right)\right\} < 0, \quad (36)$$

with $p_{\inf} = \inf\left\{p\left(x\right) : 0 \leq x \leq 1\right\} > 0$ and $p_{\sup} = \sup\left\{p\left(x\right) : 0 \leq x \leq 1\right\} < 1$. Note that these results are consistent with what has been shown in the case of constant probabilities by Mitra et al. (2003) for intermediate values of the constant probability p (*i.e.*, $1/3 \leq p \leq 2/3$) and by Shmerkin (2014) for smaller and larger values (*i.e.*, $p < 1/3$ and $p > 2/3$). Therefore, the capital share plays an important role in the determination of the steady state of our state-dependent-probability extended Brock and Mirman's (1972) model as its magnitude drives the singularity vs. absolute continuity properties of the invariant distribution, and, as we are going to see through some specific examples, different values of the capital share have important implications for the long run macroeconomic dynamics.

Also for the affine IFSSDPs (31) with associated state-dependent probabilities (32)–(35) it is impossible to explicitly characterize their invariant measure; once again we rely on their numerical approximation as the marginal measure obtained after some iterations of the Markov operator (39). We rely on the same parametrization earlier employed, but we consider different values of the capital share in order to understand how this parameter affects our previous conclusions. Specifically, we consider the following values for the capital share:

$$\alpha = 0.33, \qquad \alpha = 0.5, \qquad \text{and} \qquad \alpha = 0.8. \quad (37)$$

For $\alpha = 0.5$ we keep the same ε value as in Sections 3 and 4: $\varepsilon = 0.1756$. Hence, the first-best optimal policies (17) and (24) are characterized by parameters $\theta_1 = 0.7137$ and $\theta_2 = 0.2463$ respectively, while the second-best policy (11) is defined by $\theta_3 = 0.48$. According to Proposition 4, to the original nonlinear probabilities $p_1\left(k\right) = 0.01 - (0.1756) \ln k$ and $p_2\left(k\right) = 0.99 + (0.1756) \ln k$ for the first-best dynamics correspond the log-linearized affine probabilities (32) and (33) defined as $p_{FD}\left(x\right) = 0.6154 - (0.4870)\, x$ and $p_{FI}\left(x\right) = 0.0107 + (0.4870)\, x$ respectively, while for the second-best dynamics correspond the log-linearized affine probabilities (34) and (35) defined as $p_{SD}\left(x\right) = 0.7548 - (0.4870)\, x$ and $p_{SI}\left(x\right) = 0.2452 + (0.4870)\, x$ respectively. Figure 4 shows the first $n = 7$ iterations of our Maple algorithm for the four cases just described, starting from the uniform initial distribution, $\mu_0\left(x\right) \equiv 1$ over the interval $[0, 1]$. Note that for $\alpha = 0.5$ the support of the invariant measure $\bar{\mu}$ is always the full interval $[0, 1]$.

By comparing Figures 2(c), 2(f), 3(c) and 3(f)—plotting the approximations of the invariant measure of the dynamics defined by the original nonlinear maps—with Figure 4 it is apparent that the main features of the invariant measures are preserved under the log-linear transformation, both for the maps through (30) and for the state-dependent probabilities according to (32)–(35), in all four scenarios: when the probability is decreasing (Figures 2(c), 3(c), 4(a) and 4(c)) the invariant measure $\bar{\mu}$ concentrates most of the mass on x values close to the endpoints of the support; while when the probability is increasing (Figures 2(f), 3(f), 4(b) and 4(d)) the invariant measure $\bar{\mu}$ tends to spread most of the mass on x values more evenly in the middle of the support. Consistent with Figure 2(c) vs. Figure 3(c) and 2(f) vs. Figure 3(f), the invariant measure generated by first-best dynamics (top panels in Figure 4) concentrates more mass close to the right endpoint, 1, of the support $[0, 1]$ than what the invariant measure associated to second-best dynamics (bottom panels in Figure 4) does. In Figure 4 this feature is more neatly observable than in Figures 2 and 3 because in the former the support, the interval $[0, 1]$, is the same in all four scenarios, while the comparison between Figures 2 and 3 is somewhat more problematic due to (relevant) differences in the diameters of the supports of the invariant measures. This confirms our previous conclusions regarding the inefficiency in a decentralized setting: independent of whether the state-dependent probability is increasing or decreasing with capital, decentralization leads to a stochastic steady steady state characterized on average by lower capital levels.

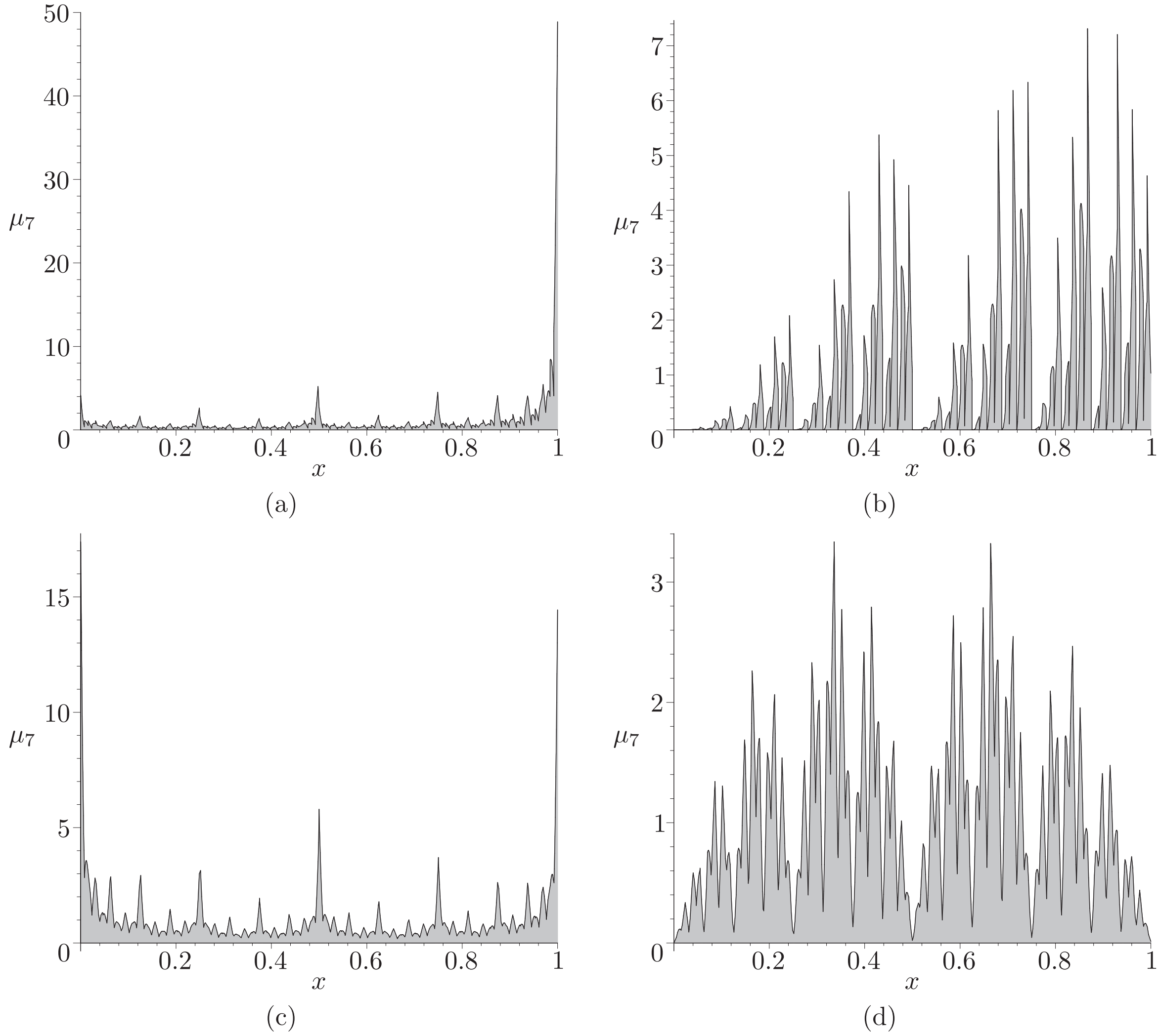
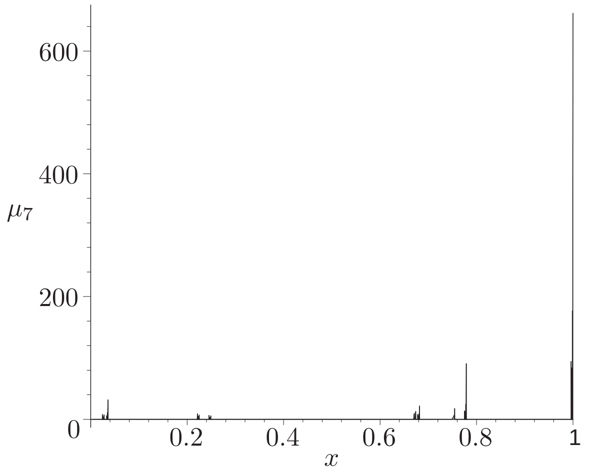


FIGURE 4: 7^{th} iteration of our Algorithm to approximate the Markov operator (39) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.5$, $\varepsilon = 0.1756$ with a) $\theta_1 = 0.7137$ (first-best dynamics) and decreasing probability $p_{FD}(x) = 0.6154 - (0.4870)\,x$; b) $\theta_2 = 0.2463$ (first-best dynamics) and increasing probability $p_{FI}(x) = 0.0107 + (0.4870)\,x$; c) $\theta_3 = 0.48$ (second-best dynamics) and decreasing probability $p_{SD}(x) = 0.7548 - (0.4870)\,x$; d) $\theta_3 = 0.48$ (second-best dynamics) and increasing probability $p_{SI}(x) = 0.2452 + (0.4870)\,x$.

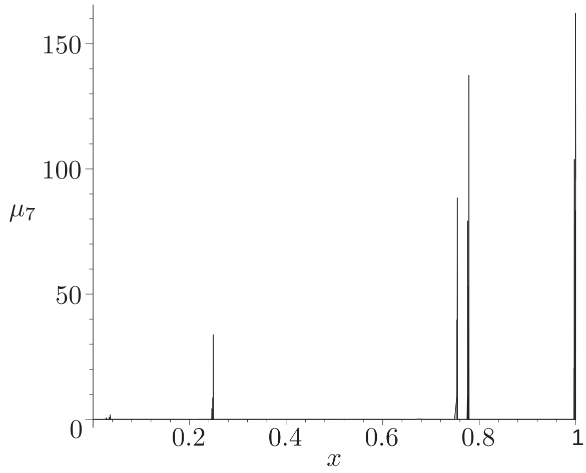
Finally, a direct comparison between the invariant measure associated to each nonlinear dynamic and its log-linearized affine counterpart introduces a slight 'uniform' shift of the mass to the right in all four cases: while this is self evident between Figures 2(f) vs. 4(b), Figures 3(c) vs. 4(c) and Figures 3(f) vs. 4(d), in the case of Figures 2(c) vs. 4(a) note that, although the spike on the right in the former figure is higher than a similar spike in the latter figure, the spikes on the left in the former figure are smaller in the latter figure and, although having a lower edge, a larger mass is being accumulated close to the endpoint 1 in the latter figure. The height and irregularity of the spikes in all four plots in Figure 4 are consistent with an invariant distribution $\bar{\mu}$ which is singular with respect to Lebesgue measure, as established by point 2. of Theorem 5 in Appendix A for $\alpha = \beta = 0.5$ and $p_j(x) \neq \alpha = 0.5$ for $i = 1, 2$ and $j = F, S$.

With different values of α condition (20) may no longer be met, thus we cannot rely on the same values of ε we have employed thus far in all our simulations. Specifically, for $\alpha = 0.33$ the ε value satisfying condition (20) with equality is 0.1720, so that we set $\varepsilon = 0.1719$, that is, 0.0001 less that its upper bound. Thus, the relevant interval for the nonlinear dynamics becomes $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0033, 1]$, the first-best optimal policies (17) and (24) are characterized by parameters $\theta_1 = 0.5456$ and $\theta_2 =$0.0880 respectively, while the second-best policy (11) is defined by $\theta_3 = 0.3168$. According to Proposition 4, to the original nonlinear probabilities $p_1(k) = 0.01 - (0.1719)\ln k$ and $p_2(k) = 0.99 + (0.1719)\ln k$ for the first-best dynamics correspond the log-linearized affine probabilities (32) and (33) defined as $p_{FD}(x) = 0.5211 - (0.3557)x$ and $p_{FI}(x) = 0.0110 + (0.3557)x$ respectively, while for the second-best dynamics correspond the log-linearized affine probabilities (34) and (35) defined as $p_{SD}(x) = 0.6606 - (0.3557)x$ and $p_{SI}(x) = 0.3394 + (0.3557)x$ respectively. Figure 5 shows the first $n = 7$ iterations of our Maple algorithm for the four cases just described, starting from the uniform initial distribution, $\mu_0(x) \equiv 1$ over the interval $[0, 1]$. For $\alpha = 0.33$ the support of the invariant measure $\bar{\mu}$ turns out to be classical Ternary Cantor set. In fact, all invariant measures in Figure 5 concentrate on a much thinner and sparser set than the invariant distributions in Figure 4; besides this feature, the general patterns exhibited by the approximations of Figure 5 seem consistent with that already discussed for the approximations in Figure 4. Clearly, consistent with point 1. of Theorem 5, all four invariant measures in Figure 5 must be singular with respect to Lebesgue measure.

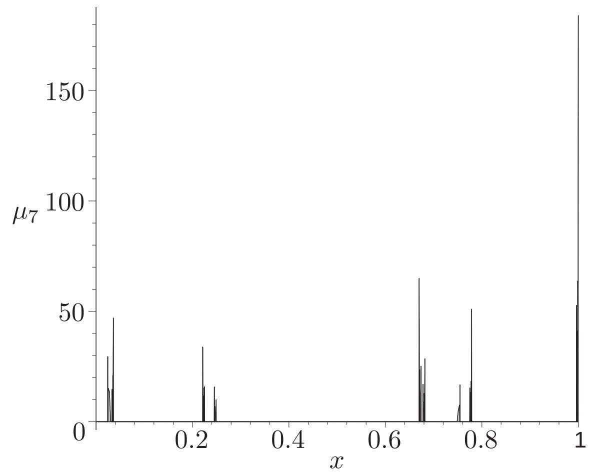
If $\alpha = 0.8$ the ε value satisfying condition (20) with equality is 0.1058, so that we set $\varepsilon = 0.1057$, that is, 0.0001 less that its upper bound. Now, the relevant interval for the nonlinear dynamics becomes $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0001, 1]$, the first-best optimal policies (17) and (24) are characterized by parameters $\theta_1 =$0.9087 and $\theta_2 =$0.6273 respectively, while the second-best policy (11) is defined by $\theta_3 = 0.768$. According to Proposition 4, to the original nonlinear probabilities $p_1(k) = 0.01 - (0.1057)\ln k$ and $p_2(k) = 0.99 + (0.1057)\ln k$ for the first-best dynamics correspond the log-linearized affine probabilities (32) and (33) defined as $p_{FD}(x) = 0.7932 - (0.7326)x$ and $p_{FI}(x) =$0.0110 $+ (0.7326)x$ respectively, while for the second-best dynamics correspond the log-linearized affine probabilities (34) and (35) defined as $p_{SD}(x) = 0.8821 - (0.7326)x$ and $p_{SI}(x) = 0.1179 + (0.7326)x$ respectively. Figure 6 shows the first $n = 7$ iterations of our Maple algorithm for the four cases just described, starting from the uniform initial distribution, $\mu_0(x) \equiv 1$ over the interval $[0, 1]$. For $\alpha = 0.8$ the support of the invariant measure $\bar{\mu}$ is the full interval $[0, 1]$. Again, the general patterns exhibited by the approximations of Figure 6 are consistent with those already discussed for the approximations in Figures 4 and 5. In this case the two maps $w_l(x) = \alpha x$ and $w_h(x) = \alpha x + (1 - \alpha)$ in the IFSSDP (31) exhibit a large overlapping region, with magnitude of 0.6.



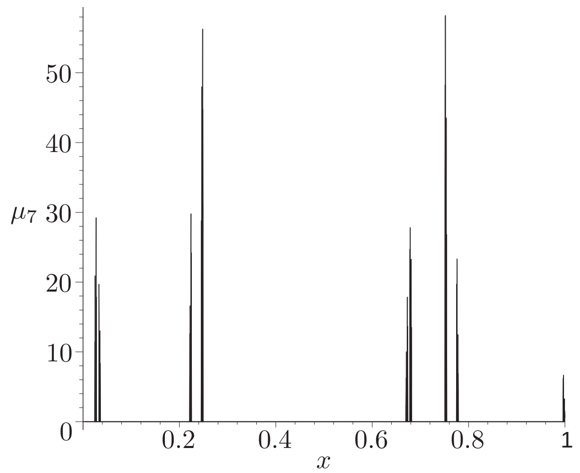
(a)



(b)



(c)



(d)

FIGURE 5: 7^{th} iteration of our Algorithm to approximate the Markov operator (39) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.33$, $\varepsilon = 0.1719$ with a) $\theta_1 = 0.5456$ (first-best dynamics) and decreasing probability $p_{FD}(x) = 0.5211 - (0.3557)\,x$; b) $\theta_2 = 0.0880$ (first-best dynamics) and increasing probability $p_{FI}(x) = 0.0110 + (0.3557)\,x$; c) $\theta_3 = 0.3168$ (second-best dynamics) and decreasing probability $p_{SD}(x) = 0.6606 - (0.3557)\,x$; d) $\theta_3 = 0.3168$ (second-best dynamics) and increasing probability $p_{SI}(x) = 0.3394 + (0.3557)\,x$.

Such a property implies that the invariant distribution $\bar{\mu}$ is more likely to be smooth, a feature clearly apparent from all plots in Figures 6. More precisely, the invariant measures approximated in Figures 6(a), 6(c) and 6(d) satisfy the condition $\alpha > e^{\Theta}$, and thus they are *almost surely* absolutely continuous. In fact, for $p_{FD}(x) = 0.7932 - (0.7326)\,x$ in Figure 6(a) the term Θ in (36) turns out to be $\Theta = -0.2287$, so that $e^{\Theta} = e^{-0.2287} = 0.7956 < 0.8 = \alpha$; similarly, for both $p_{SD}(x) = 0.8821 - (0.7326)\,x$ and $p_{SI}(x) = 0.1179 + (0.7326)\,x$ in Figures 6(c) and 6(d) the term Θ turns out to be the same, $\Theta = -0.2484$, so that $e^{\Theta} = e^{-0.2484} = 0.78 < 0.8 = \alpha$. In other words, the spikes present in the finite-time approximation of $\bar{\mu}$ in Figures 6(a), 6(c) and 6(d) are likely to be asymptotically smoothed out as the number of iterations approaches infinity. Conversely, the condition $\alpha > e^{\Theta}$ does not hold for the invariant measure approximated in Figure 6(b), as for $p_{FI}(x) = 0.0110 + (0.7326)\,x$ the term is $\Theta = -0.0607$, so that $e^{\Theta} = e^{-0.0607} = 0.9411 > 0.8 = \alpha$.

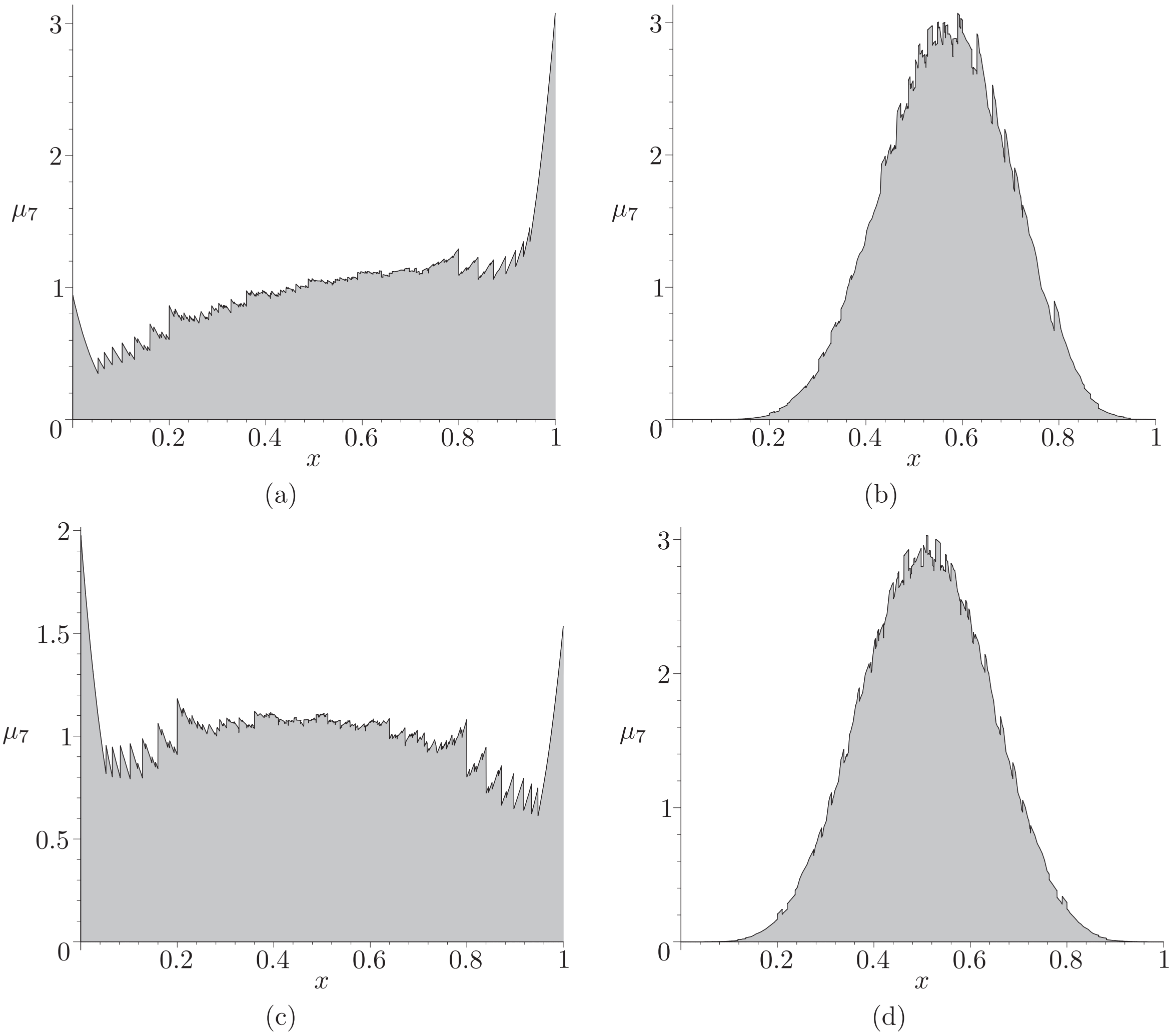


FIGURE 6: 7^{th} iteration of our Algorithm to approximate the Markov operator (39) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.8$, $\varepsilon = 0.1057$ with a) $\theta_1 = 0.9087$ (first-best dynamics) and decreasing probability $p_{FD}(x) = 0.7932 - (0.7326)\,x$; b) $\theta_2 = 0.6273$ (first-best dynamics) and increasing probability $p_{FI}(x) = 0.0110 + (0.7326)\,x$; c) $\theta_3 = 0.768$ (second-best dynamics) and decreasing probability $p_{SD}(x) = 0.8821 - (0.7326)\,x$; d) $\theta_3 = 0.768$ (second-best dynamics) and increasing probability $p_{SI}(x) = 0.1179 + (0.7326)\,x$.

All our numerical examples confirm our previous results regarding the effects of the property of increasingness or decreasingness of the state-dependent probabilities on the steady state capital distribution, along with those regarding the inefficiency arising in a decentralized setting. This also suggests that our results are robust to different values of the capital share, and in particular different values of the parameter only determine whether the invariant measure may turn out to be singular or absolutely continuous, without modifying the nature of our main conclusions.

6 Conclusion

We extend the classical discrete time stochastic one-sector growth model with logarithmic utility and Cobb-Douglas production function á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. Under state-dependent probabilities the probability of occurrence of a given shock depends on the capital stock, thus as the economy accumulates more capital along its process of economic development the probability of occurrence of different shocks changes over time. As the social planner in making their investment decisions needs to account for how the future capital stock level will impact these probabilities, the optimal policy critically depends on the characteristics of the state-dependent probability function. Therefore, state-dependent probabilities act as an engine of capital accumulation, which through its effects on the probability of shocks realization impacts the process of economic development. We show that whenever the probability (assumed to take a logarithmic form) is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, and this incentivizes the planner to increase (decrease) his capital investment, generating monotonic (non-monotonic) economic dynamics, resulting in a steady state capital distribution more skewed (symmetric) towards the upper extreme (around the middle) of its support. In a decentralized setting in which single individuals do not internalize the effects of capital accumulation on the state-dependent probabilities, capital accumulation turns out to be independent of the probabilities leading to situations of underinvestment (overinvestment) with respect the first-best. We also show that both the centralized and decentralized optimal solutions can be converted into a contractive affine IFS with affine SDP which, under rather general conditions, converges to an invariant self-similar measure supported on a (possibly fractal) compact attractor.

To the best of our knowledge, ours is the first attempt to analyze the role of state-dependent probabilities in optimal stochastic economic growth settings. Therefore, several interesting issues associated with the role of state-dependent probabilities on macroeconomic dynamics still need to be uncovered. We have considered only the situation in which the probability function monotonically depends on the capital stock, thus it is natural to wonder how results may change in more general settings in which the probability may be non-monotonic in capital. Moreover, our focus has been placed on the effects of state-dependent probabilities on capital accumulation, while other macroeconomic variables such as wealth and public debt may be affected by shocks occurring with state-dependent probabilities as well, thus it would be interesting to analyze the consequence of state-dependent probabilities on other macroeconomic dynamics. The analysis of these further issues is left for future research.

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A Iterated Function Systems

We now review some basic concepts and the main results in the theory of Iterated Function Systems (IFSs) with constant and state-dependent probabilities. The notion of IFS was firstly introduced by Barnsley et al. (1990) and Hutchinson (1981) and then extended in different contexts (see Kunze et al., 2012, and the references therein).

Given a compact metric space (X, d), an N-map *Iterated Function System* (IFS) on X, $\mathbf{w} = \{w_1, \ldots, w_N\}$, is a set of N contraction mappings on X, *i.e.*, $w_i : X \to X$, $i = 1, \ldots, N$, with contraction factors $c_i \in [0, 1)$. It can be proved that under these assumptions the following set-valued mapping $\hat{\mathbf{w}}$ defined on the space $\mathcal{H}(X)$ of nonempty compact subsets of X:

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{H}(X).$$

is a contraction on the complete metric space $\mathcal{H}(X)$ endowed with the classical Hausdorff distance h defined as:

$$h(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{x\in B}\inf_{y\in A} d(x,y)\right\}.$$

This result implies the existence and uniqueness of a fixed point A such that $\hat{\mathbf{w}}(A) = A$. Moreover, A is self-similar, that is, it is the union of distorted copies of itself and it is also attracting, that is, for any $B \in \mathcal{H}(X)$, $h(A, \hat{\mathbf{w}}^t B) \to 0$ as $t \to \infty$.

An N-map *iterated function system with (constant) probabilities* $(\mathbf{w}, \mathbf{p})$ is an N-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \ldots, p_N\}$, $\sum_{i=1}^{N} p_i = 1$. It can be proved that the Markov operator defined by $\nu(S) = (M\mu)(S)$:

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \mu\left(w_i^{-1}(S)\right).$$

is a contraction mapping on the space $\mathcal{M}(X)$ composed by all probability measures on (Borel subsets of) X with respect to the *Monge-Kantorovich* distance defined as follows: For any pair of probability measures $\mu, \nu \in \mathcal{M}(X)$, we have

$$d_{MK}(\mu,\nu) = \sup_{f\in Lip_1(X)} \left[\int f d\mu - \int f d\nu\right],$$

where $Lip_1(X) = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq d(x,y)\}$. These assumptions imply the existence of a unique attracting measure $\bar{\mu} \in \mathcal{M}(X)$.

The family of IFS with state-dependent probabilities extends the above definitions. Within this framework, the probabilities p_i are no longer constant but they are are state-dependent, *i.e.*, $p_i : X \to [0,1]$ such that:

$$\sum_{i=1}^{N} p_i(x) = 1, \qquad \text{for all } x \in X. \tag{38}$$

The result is an N-map *IFS with state-dependent probabilities* (IFSSDP). The Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ associated with an N-map IFSSDP, $(\mathbf{w}, \mathbf{p})$, is defined as:

$$\nu(S) = M\mu(S) = \sum_i \int_{w_i^{-1}(S)} p_i(x)\, d\mu(x), \tag{39}$$

where $\mu \in \mathcal{M}(X)$ and $S \subset X$ is a Borel set.

Theorem 1 (La Torre et al., 2018a) *Given M as defined in equation (39), then M maps $\mathcal{M}(X)$ to itself. In other words, if $\mu \in \mathcal{M}(X)$, then $\nu = M\mu \in \mathcal{M}(X)$.*

Under appropriate conditions, the above Markov operator can be contractive with respect to the Monge-Kantorovich metric.

Theorem 2 (La Torre et al., 2018a) *Let* (X, d) *be a compact metric space and* $(\mathbf{w}, \mathbf{p})$ *an* N*-map IFSSDP with IFS maps* $w_i : X \to X$ *with contraction factors* $c_i \in [0, 1)$*. Furthermore, assume that the probabilities* $p_i : X \to \mathbb{R}$ *are Lipschitz functions, with Lipschitz constants* $K_i \geq 0$*. Let* $M : \mathcal{M}(X) \to \mathcal{M}(X)$ *be the Markov operator associated with this IFSSDP, as defined in (39). Then for any* $\mu, \nu \in \mathcal{M}(X)$*,*

$$d_{MK}(M\mu, M\nu) \leq (c + KDN)\, d_{MK}(\mu, \nu),$$

where $c = \max_i c_i$*,* $K = \max_i K_i$ *and* $D = \mathrm{diam}(X) < \infty$*.*

Theorem 3 (La Torre et al., 2018a) *Under the same assumptions as in the above Theorem, if* $c + KDN < 1$ *then the Markov operator* M *has a unique fixed point* μ *in* $\mathcal{M}(X)$*. Furthermore, for any* $\nu \in \mathcal{M}(X)$*, the orbit* $M^n\nu$ *converges to* μ *in* d_{MK} *when* $n \to +\infty$*.*

We now describe the so-called *Chaos Game* for an IFS with probabilities. Start with $x_0 \in X$, and define the sequence $x_t \in X$ by:

$$x_{t+1} = w_{\sigma_t}(x_t),$$

where $\sigma_t \in \{1, 2, \ldots, N\}$ is chosen according to the probabilities $p_i(x_t)$ (that is, $P[\sigma_t = i] = p_i(x_t)$). We note that the sequence (x_t) is a Markov chain with values in X. The following theorem (from results in Elton, 1987; and Barnsley et al., 1988) gives conditions as to when an IFSSDP has a unique stationary distribution μ and the Chaos Game "converges" to μ in a distributional sense.

Theorem 4 (Elton, 1987; Barnsley et al., 1988) *Suppose that there is a* $\delta > 0$ *so that* $p_i(x) > \delta$ *for all* $x \in X$ *and* $i = 1, 2, \ldots, N$ *and suppose further that the moduli of continuity of the* p_i*s satisfy Dini's condition (see Elton, 1987; and Barnsley et al., 1988). Then there is a unique stationary distribution* $\bar{\mu}$ *for the Markov operator. Furthermore, for each continuous function* $f : X \to \mathbb{R}$*,*

$$\frac{1}{t+1}\sum_{i=0}^{t} f(x_i) \to \int_X f(x)\ d\bar{\mu}(x). \tag{40}$$

Theorem 4 can be used to show the following result.

Corollary 1 *Suppose that the IFSSDP* $\{\mathbf{w}, p_i\}$ *satisfies the hypothesis of Theorem 4. Then the support of the invariant measure* $\bar{\mu}$ *of the* N*-map IFSSDP* $(\mathbf{w}, \mathbf{p})$ *is the attractor* A *of the IFS* $\mathbf{w}$*, i.e.,*

$$\mathrm{supp}\,\bar{\mu} = A.$$

Therefore the invariant measure μ satisfies the following equation

$$\mu(S) = \sum_i \int_{w_i^{-1}(S)} p_i(x)\, d\mu(x), \tag{41}$$

for any subset S of X. This equation shows how the invariant measure can be obtained by combining different distorted copies of itself. This justifies why the invariant measure is a self-similar object. Moreover, the invariant measure can be characterized by either singularity or absolute continuity, and such a property can be determined as follows.

Theorem 5 (La Torre et al., 2023) *Take the two-map IFS on $\mathcal{R}$ given by $\{\alpha x + \tau_1, \beta x + \tau_2\}$, with $\alpha, \beta \in [0, 1)$ along with the two probability functions $p_1(x) = p(x)$ and $p_2(x) = 1 - p(x)$. Assume that $\delta < p(x) < 1 - \delta$ for all x and some $\delta > 0$ and also that p is Hölder continuous. Let $\mu_{\alpha,\beta}$ be the invariant measure of this state-dependent IFS.*

1. *If $0 \leq \alpha + \beta < 1$ then $\mu_{\alpha,\beta}$ is singular with respect to Lebesgue measure.*
2. *If $\alpha + \beta = 1$ then $\mu_{\alpha,\beta}$ is either singular with respect to Lebesgue measure or is equal to the (normalized) Lebesgue measure on the closed interval with endpoints $\frac{\tau_1}{1-\alpha}$ and $\frac{\tau_2}{1-\beta}$ and $p(x) = \alpha$.*
3. *For each $\alpha + \beta > 1$, let $h_{\alpha,\beta}$ be defined by*

$$h_{\alpha,\beta} = -\int \{p(x) \ln[p(x)] + [1 - p(x)] \ln[1 - p(x)]\} d\mu_{\alpha,\beta}(x)$$

and

$$\chi_{\alpha,\beta} = -\log(\beta) + [\log(\beta) - \log(\alpha)] \int p(x) \ d\mu_{\alpha,\beta}(x).$$

Then $\mu_{\alpha,\beta}$ is singular for every α, β with $h_{\alpha,\beta} < \chi_{\alpha,\beta}$.

Furthermore, there is an open subset $\Theta \subset \{(\alpha, \beta) \in (0, 1)^2 : \alpha + \beta > 1\}$ so that $\mu_{\alpha,\beta}$ is absolutely continuous with respect to Lebesgue measure for Lebesgue almost every $(\alpha, \beta) \in \Theta$ such that $h_{\alpha,\beta} > \chi_{\alpha,\beta}$.

These basic concepts related to the theory of IFSSDP are useful to derive the steady state equilibrium and understand its characteristics in our stochastic economic growth model.

B Proofs of the Main Results

B.1 Proof of Lemma 1

Fix an initial value k_0 for capital, possibly, but not necessarily, such that $k_0 \in \left(0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$, and suppose, by contradiction, that the optimal saving/investment $y^* = k_{t+1}$ in each period t remains bounded inside the interval $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$. Hence, both definitions (3) and (4) imply that at each time $t \geq 0$ the probability of the shock r is constant—given by $p(y^*) \equiv 1 - \delta$ or $p(y^*) \equiv \delta$ respectively—so that either the Bellman equation (7) or (9) fully represent problem (2). It is well known that the optimal policy solving either equation (7) or equation(9) is the same and is given by (11); such a policy generates trajectories $k_{t+1} = h(k_t, z_t) = \alpha\beta z_t k_t^\alpha$ having the deterministic trajectory generated by the lower map $k_{t+1} = \alpha\beta r k_t^\alpha$ as lower bound, so that $k_{t+1} = \alpha\beta z_t k_t^\alpha \geq \alpha\beta r k_t^\alpha$ for all $t \geq 0$. The lower bound trajectory generated by $\underline{k}_{t+1} = \alpha\beta r \underline{k}_t^\alpha$ converges to the (deterministic) fixed point $\lim_{t\to\infty} \alpha\beta r \underline{k}_t^\alpha = (\alpha\beta r)^{\frac{1}{1-\alpha}}$. As condition (12) is equivalent to

$$e^{-\frac{1-\delta-\gamma}{\varepsilon}} < (\alpha\beta r)^{\frac{1}{1-\alpha}},$$

we conclude that there exists a finite date $\tau \geq 0$ such that $k_\tau \geq \underline{k}_\tau \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, thus contradicting the assumption that k_t remains bounded inside the interval $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ for all $t \geq 0$.

B.2 Proof of Proposition 1

Under the assumption that $B - \varepsilon C \ln r > 0$ (we shall see that it holds at the end of the proof) the RHS in (13) is strictly concave in y, and the FOC with respect to y yields the unique solution

$$y^* = \frac{\beta (B - \varepsilon C \ln r)}{1 + \beta (B - \varepsilon C \ln r)} z k^\alpha, \tag{42}$$

Substituting y^* as in (42) into the RHS of (13) after some algebra yields

$$\begin{aligned} V(k,z) &= A + B \ln k + C \ln z \\ &= \ln \left[z k^\alpha - \frac{\beta (B - \varepsilon C \ln r)}{1 + \beta (B - \varepsilon C \ln r)} z k^\alpha \right] + \beta (B - \varepsilon C \ln r) \ln \left[\frac{\beta (B - \varepsilon C \ln r)}{1 + \beta (B - \varepsilon C \ln r)} z k^\alpha \right] \\ &\quad + \beta (A + \gamma C \ln r) \\ &= \alpha [1 + \beta (B - \varepsilon C \ln r)] \ln k + [1 + \beta (B - \varepsilon C \ln r)] \ln z \\ &\quad + \beta (B - \varepsilon C \ln r) \ln [\beta (B - \varepsilon C \ln r)] - [1 + \beta (B - \varepsilon C \ln r)] \ln [1 + \beta (B - \varepsilon C \ln r)] \\ &\quad + \beta (A + \gamma C \ln r). \end{aligned}$$

By equating all similar terms in both sides we find that a solution of the Bellman equation (13) is given by the constants A, B and C that satisfy

$$\begin{cases} (1-\beta) A = \beta \gamma C \ln r + \beta (B - \varepsilon C \ln r) \ln [\beta (B - \varepsilon C \ln r)] \\ \qquad\qquad - [1 + \beta (B - \varepsilon C \ln r)] \ln [1 + \beta (B - \varepsilon C \ln r)] \\ B = \alpha [1 + \beta (B - \varepsilon C \ln r)] \\ C = 1 + \beta (B - \varepsilon C \ln r). \end{cases}$$

From the second and third equations we see that $B = \alpha C$, so that, after substituting this in the third equation, we easily find the value of C as in (16), $C = \frac{1}{1-\beta(\alpha - \varepsilon \ln r)}$, which, when replaced into the second equation, yields the (crucial) value for B as in (15): $B = \frac{\alpha}{1-\beta(\alpha-\varepsilon \ln r)}$.

After cumbersome algebra the value of parameter A can be easily obtained; as $B = \alpha C = \frac{\alpha}{1-\beta(\alpha-\varepsilon \ln r)}$, we get

$$\beta (B - \varepsilon C \ln r) = \beta (\alpha C - \varepsilon C \ln r) = \beta (\alpha - \varepsilon \ln r) C = \frac{\beta (\alpha - \varepsilon \ln r)}{1 - \beta (\alpha - \varepsilon \ln r)},$$

so that:

$$\begin{aligned} A &= \frac{1}{1-\beta} \left\{ \frac{\beta (\alpha - \varepsilon \ln r)}{1 - \beta (\alpha - \varepsilon \ln r)} \ln \left[\frac{\beta (\alpha - \varepsilon \ln r)}{1 - \beta (\alpha - \varepsilon \ln r)} \right] \right. \\ &\quad \left. - \left[1 + \frac{\beta (\alpha - \varepsilon \ln r)}{1 - \beta (\alpha - \varepsilon \ln r)} \right] \ln \left[1 + \frac{\beta (\alpha - \varepsilon \ln r)}{1 - \beta (\alpha - \varepsilon \ln r)} \right] + \frac{\beta \gamma \ln r}{1 - \beta (\alpha - \varepsilon \ln r)} \right\} \\ &= \frac{[1 - \beta (\alpha - \varepsilon \ln r)] \ln [1 - \beta (\alpha - \varepsilon \ln r)] + \beta (\alpha - \varepsilon \ln r) \ln [\beta (\alpha - \varepsilon \ln r)] + \beta \gamma \ln r}{(1-\beta) [1 - \beta (\alpha - \varepsilon \ln r)]}, \end{aligned}$$

which is the expression in (14).

After replacing B and C as in (15) and in (16) respectively into (42) we easily obtain

$$y^* = h(k,z) = \frac{\beta (B - \varepsilon C \ln r)}{1 + \beta (B - \varepsilon C \ln r)} z k^\alpha = \beta (\alpha - \varepsilon \ln r) z k^\alpha,$$

which confirms the expression in (17) for the optimal policy.

The solution in (17) is certainly interior under condition (12). In fact, on one hand it is straightforward to show that

$$\varepsilon < -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)} \qquad \Longleftrightarrow \qquad \alpha\beta r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

which implies that, as $-\varepsilon \ln r > 0$, for any $k \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$,

$$y^* = \beta(\alpha - \varepsilon \ln r) z k^{\alpha} > \alpha\beta z k^{\alpha} \geq \alpha\beta r k^{\alpha} \geq \alpha\beta r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

that is, $y^* > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. On the other hand, to prove that $y^* = \beta(\alpha - \varepsilon \ln r) z k^{\alpha} < z k^{\alpha}$ we show that condition (12) implies that $0 < \beta(\alpha - \varepsilon \ln r) < \alpha - \varepsilon \ln r < 1$. To this purpose note that, as $0 < \alpha\beta < 1$ and $0 < r < 1$, the following holds:

$$\ln(\alpha\beta) + \ln r = \ln(\alpha\beta r) < \ln r \quad \Longleftrightarrow \quad -\frac{1-\alpha}{\ln r} > -\frac{1-\alpha}{\ln(\alpha\beta r)}$$
$$\Longrightarrow \quad -\frac{1-\alpha}{\ln r} > -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)} > \varepsilon,$$

where the last inequality is condition (12); as the last two inequalities are equivalent to $\alpha - \varepsilon \ln r < 1$, we have just established that that $y^* < z k^{\alpha}$.

The property that $0 < \beta(\alpha - \varepsilon \ln r) < 1 \Longleftrightarrow 1 - \beta(\alpha - \varepsilon \ln r) > 0$ also implies that both coefficients B and C are strictly positive; this establishes that the RHS in (13) is strictly concave.

Finally, it is a simple exercise to show that problem (2) satisfies all assumptions of Theorem 9.12 on p. 274 in Stokey and Lucas (1989): therefore, the function $V(k,z) = A + B\ln k + C\ln z$—with coefficients A, B and C defined in (14), (15) and (16) respectively—that solves the Bellman equation (13) is exactly the value function of problem (2), while the function $h(k_t^*, z_t) = \beta(\alpha - \varepsilon \ln r) z_t (k_t^*)^{\alpha}$ defined in (17) is exactly the optimal policy. We omit the details for brevity.

B.3 Proof of Proposition 2

Provided that $B + \varepsilon C \ln r > 0$, the RHS in (19) is strictly concave in y; hence, steps similar to those used in the proof of Proposition 1 easily yield the values A, B and C as in (21), (22) and (23), together with the optimal policy as in (24) and the optimal consumption as in (25). Being the same exercise as in the previous proof of Proposition 1, also establishing that Theorem 9.12 on p. 274 in Stokey and Lucas (1989) holds is straightforward, so that the function $V(k,z) = A + B\ln k + C\ln z$—with coefficients A, B and C defined in (21), (22) and (23) respectively—that solves the Bellman equation (19) is exactly the value function of problem (2), while the function $h(k_t^*, z_t) = \beta(\alpha + \varepsilon \ln r) z_t (k_t^*)^{\alpha}$ defined in (24) is exactly the optimal policy.

We only need to establish that the unique solution in (24) is interior under condition (20). In fact, on one hand it is immediately shown that

$$e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\varepsilon < \alpha\beta r \qquad \Longleftrightarrow \qquad \beta(\alpha + \varepsilon \ln r) r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

which implies that, for any $k \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$,

$$y^* = \beta(\alpha + \varepsilon \ln r) z k^{\alpha} \geq \beta(\alpha + \varepsilon \ln r) r k^{\alpha} \geq \beta(\alpha + \varepsilon \ln r) r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

that is, $y^* > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. Note, in turn, that, as $e^{-\frac{1-\delta-\gamma}{\varepsilon}} > 0$, the last inequality also establishes that $\beta(\alpha + \varepsilon \ln r) > 0$. On the other hand, as $\varepsilon \ln r < 0$, $\beta(\alpha + \varepsilon \ln r) < 1$ definitely holds, so that $y^* = \beta(\alpha + \varepsilon \ln r) zk^\alpha < zk^\alpha$ as well.

Finally,

$$\beta(\alpha + \varepsilon \ln r) > 0 \quad \Longleftrightarrow \quad \frac{\beta\alpha}{1 - \beta(\alpha + \varepsilon \ln r)} + \frac{\beta\varepsilon \ln r}{1 - \beta(\alpha + \varepsilon \ln r)} = \beta(B + \varepsilon C \ln r) > 0,$$

which establishes that the RHS in (19) is strictly concave in y.

B.4 Proof of Proposition 4

The inverse transformation of (30) yields k as a function of x according to $k = (\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x$. Therefore, $p_j(x) = p(k) = p\left[(\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, so that, according to (3), $p_{FD}(x) = \gamma - \varepsilon \ln k = \gamma - \varepsilon \ln\left[(\theta_1 r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, which is equivalent to (32), while, according to (4), $p_{FI}(x) = 1 - \gamma + \varepsilon \ln k = 1 - \gamma + \varepsilon \ln\left[(\theta_2 r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, which is equivalent to (33). A similar argument holds for the affine probabilities in (34) and (35) for $j = SD, SI$ and $\theta_3 = \alpha\beta$. As, under Assumptions A.1 and A.2, Propositions 1 and 2 establish that $k_t \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$ for all $t \geq 0$, definitions (3) and (4) guarantee that $p_j(x) = p(k) = p\left[(\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$ satisfy $0 < p_j(x) < 1$ for all $x \in [0, 1]$ and $j = FD, FI, SD, SI$. Finally, as $\frac{\varepsilon \ln r}{1-\alpha} < 0$, clearly $p_{FD}(x)$ and $p_{SD}(x)$ are decreasing, while, as $-\frac{\varepsilon \ln r}{1-\alpha} > 0$, $p_{FI}(x)$ and $p_{SI}(x)$ are increasing.

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