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SIMULATING STOCK MARKETS: RISK-AVERSION AS A COGNITIVE BIAS

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Simulating Stock Markets: Risk-Aversion as a Cognitive Bias

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Abstract

This paper analyzes market implications of behavioral finance by means of a representative agent model of financial market. The goal is to provide a model as tool for studying the emergence of behavioral market anomalies. We aim to show that such model can contribute to behavioral finance research by demonstrating if and to what extent risk-aversion can be used as a substitute of individual biases in determining market anomalies.

1 Introduction

Economics and financial theories have for long been dominated by the Efficient Markets Hypothesis (EMH), which posits that market prices fully reflect all available information. Efficient markets do not allow investors to earn above-average returns without accepting above-average risks. Financial theories and models rests on a formal representation of an individual who acts as a utility maximize, given his preferences, and adheres to the axioms of a rational choice theory. Over the past decades, however, psychologists and behavioral scientists have documented robust and systematic violations of principles of expected utility theory, Bayesian learning, and rational expectations. The idea of individual investors who are prone to biases in judgment, and use various heuristics, which might lead to anomalies on the market level, has been explored within the field of behavioral finance. A number of behavioral models have been developed for the purpose of studying agents behavior, price discovery mechanisms, and the reproduction of the market anomalies.

The aim of this work is to study and analyze if and to what extent, under clear assumptions, the risk-aversion of the agent, solely, can determine deviations of prices from its fundamental value and produce the emergence of any known market anomalies,

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such as momentum and returns autocorrelations (Shiller 1981; Poterba and Summers 1988). Since the influence of the agents' behavior on the market dynamics represents a key issue of the *behavioral finance* as a whole, in the literature there are numerous theoretical studies (Barberis et al. 1998; Daniel et al. 1998; Levy et al. 1994; De Long et al. 1990; Hong and Stein 1999). A pillar of such studies is represented by the work of Barberis et al. (1998); their *Model of Investor Sentiment* tries to explain, in a relative simple way, the emergence of the empirical findings on overreaction and underreaction (Bondt and Thaler 1985; Jegadeesh and Titman 1993; Lakonishok et al. 1994). They present a model with a representative agent, showing on the theoretical ground to what extent such market anomalies are determined by a biased behavior of the market participant, in particular they refer to two specific cognitive biases: the *representativeness* [1] and the *conservatism* [2] , which have been broadly studied in the literature (Kahneman and Tversky 1974; Rabin 1998; Edwards 1968; Shefrin 2000). Although both this and Barberis et al. (1998) works pursue a similar goal, the authors present a model with a representative, risk-neutral investor with fixed discount rate, whereas this model can be seen as its reinterpretation in a representative, risk-averse framework. Barberis et al. (1998) in their model consider only one security, which pays out 100% of its earnings as dividends and they assume that the agent forms its expectation to forecast the earning stream; in our model we consider one security [3] as well but it is a short-lived asset, and we assume that agent forms its expectation to forecast the dividend stream. In both models the agents beliefs on the realization of the state of the world are determined as Markov process [4]. Furthermore, Barberis et al. (1998) consider only one case in which the agent does not realize the correct model that is actually generating earnings; he rather believe the world moves between two states or regimes, therefore he switches between two belief's models; whereas we study two different cases: one in which we assume the agent knows how actually the market behaves, and one in which the agent is 'uninformed' so that his beliefs about the realization of the state of world depend on the last observed realization.

In the following sections it will be provided a general description of the model, and then will be studied the dynamics of agents wealth and the market price of asset in the

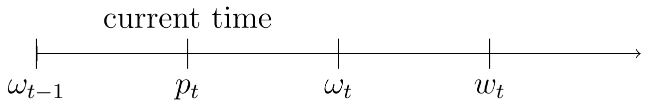
[1] Representativeness bias occurs when it is required to assess the probability of an object A belonging to a class or process B. The heuristic rule says that if object A has similar essential properties to the class B, or it reflects the salient features of the process B, then the probability of A originating from B is judged as high, and vice versa

[2] Conservatism refers to the tendency of individuals to change and adjust their beliefs only slowly in the face of new evidence/information

[3] At the macro level, we take into account also a riskless asset, that we remove afterwards to simplify the analysis, setting the riskless interest rate equal to zero

[4] A Markov process is a stochastic process that satisfies the Markov property. A process satisfies such property if one can make predictions for the future of the process based solely on its present state.

Table 1: Timing of model's evolution



two scenarios mentioned above. Finally will be shown some simulation experiments of the model and it will be discussed the implications arising from the models results.

2 A General Description of The Model

Consider a simple pure exchange economy composed by a short-lived risky asset, with price p_t, paying an amount d_t as dividends at the end of each period and a riskless asset (bond) giving in each period a constant interest rate $r > 0$. The price of riskless asset is fixed to 1. Let w_{t-1} stand for the wealth of a representative risk-averse agent at time $t-1$ and let x_t stand for the fraction of this wealth invested into the risky asset. Therefore, the agent derives the optimal amount x_t by maximizing the expected utility of his wealth at time t

$$x_t = argmax\ E[U(w_t)] = E\Big[U\Big(w_{t-1}\ x_t \frac{d_t}{p_t} + w_{t-1}(1-x_t)(1+r)\Big)\Big] \quad .$$

Assume that the world can be in two states: 1 or 2. Let ω_t be the state of the world at time t and assume further that the payoffs depends on ω_t and d_{t-1}, that is: $d_t = f_{\omega_t}(d_{t-1})$. Agent's beliefs about the states of world follow a Markov process: let Π_t be the individual probability assigned by the agent to the event that the world is in the state 1 at time t. This probability is a function of the last realization of the state of the world $\Pi_t = Prob\ [\omega_t = 1|\omega_{t-1}] = \Pi(\omega_{t-1})$.

If let $R = 1 + r$ and assume $f_2(d_{t-1}) > f_1(d_{t-1})$, the agent' s expected logarithmic utility becomes

$$\begin{aligned} (2.1)\quad \Pi_t\ \ln\Big(w_{t-1}\ x_t\frac{f_1(d_{t-1})}{p_t} + w_{t-1}(1-x_t)R\Big) + (1-\Pi_t) \\ \ln\Big(w_{t-1}\ x_t\frac{f_2(d_{t-1})}{p_t} + w_{t-1}(1-x_t)R\Big)\ . \end{aligned}$$

The first order condition is given by

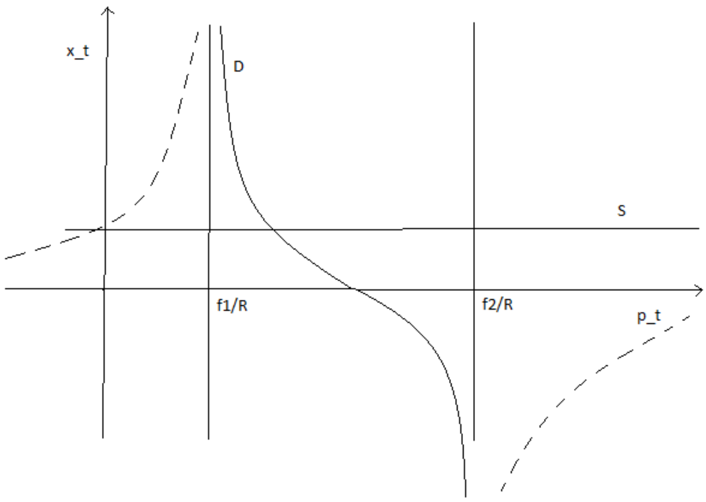


Figure 1: Demand and supply function

$$\frac{dU}{dx_t} = \frac{\Pi_t\left(\frac{f_1}{p_t} - R\right)}{x_t\frac{f_1}{p_t} + (1-x_t)R} + \frac{(1-\Pi_t)\left(\frac{f_2}{p_t} - R\right)}{x_t\frac{f_2}{p_t} + (1-x_t)R} = 0 \ ,$$

which reduces to

$$x_t\left(\frac{f_1}{p_t} - R\right)\left(\frac{f_2}{p_t} - R\right) + \frac{\Pi_t}{\frac{f_2}{p_t} - R} + \frac{(1-\Pi_t)}{\frac{f_1}{p_t} - R}\left(\frac{f_2}{p_t} - R\right) = 0 \ .$$

If $f_1/R < p < f_2/R$ then

$$x_t = p_t\left(\frac{1-\Pi_t}{p_t - \frac{f_1}{R}} - \frac{\Pi_t}{\frac{f_2}{R} - p_t}\right) \ , \tag{2.2}$$

notice that for the demand $D_t = \dfrac{x_t}{p_t}\,w_{t-1}$, it is $\dfrac{d}{dP_t}D_t < 0$, and

$$\lim_{p_t \to (f_1/R)^+} \frac{x_t}{p_t} = +\infty \ , \qquad \lim_{p_t \to (f_2/R)^-} \frac{x_t}{p_t} = -\infty \ .$$

3 Representative Agent

The price is set to the level which satisfies the market clearing conditions. If the risky asset has zero outstanding shares it is

$$p_t = \frac{1}{R}E(d_t) = \Pi_t \frac{f_1(d_{t-1})}{R} + (1-\Pi_t)\frac{f_2(d_{t-1})}{R} \ , \tag{3.1}$$

which is equivalent to the risk neutral evaluation; otherwise the price is fixed by

$$\frac{w_{t-1}\ x_t}{p_t} = 1$$

$$w_{t-1}\left(\frac{1-\Pi_t}{p_t - \frac{f_1}{R}} - \frac{\Pi_t}{\frac{f_2}{R} - p_t}\right) = 1 \ ,$$

which solving for p_t gives

$$p_t = \frac{\left(Rw_{t-1} + f_1 + f_2\right) + \sqrt{\left(Rw_{t-1} + f_1 + f_2\right)^2 - Rw_{t-1}E_{t-1}[f] - 4\ f_1 f_2}}{2\ R} \ , \tag{3.2}$$

with $E_{t-1}[f] = \Pi_{(\omega_{t-1})} f_1(d_{t-1}) + 1 - \Pi_{(\omega_{t-1})} f_2(d_{t-1})$ that is the expectation of f given all the information available at time $t-1$. In this way we obtain a quadratic equation for p_t in which from the graph is clear that only the minus solution is acceptable.

The system for wealth and prices reads

$$\begin{cases} p_t = \frac{\left(Rw_{t-1}+f_1+f_2\right)+\sqrt{\left(Rw_{t-1}+f_1+f_2\right)^2 - Rw_{t-1}E_{t-1}[f] - 4\ f_1 f_2}}{2\ R} \\ w_t = w_{t-1}\left[(x_t \frac{f_{\omega_t}(d_{t-1})}{p_t} + (1-x_t)R\right] \ . \end{cases} \tag{3.3}$$

Given the normalization condition $\frac{w_{t-1}\ x_t}{p_t} = 1$, we can rewrite di expression for the evolution of wealth

$$\begin{aligned} w_t &= \frac{w_{t-1}\ x_t f_{\omega_t}(d_{t-1})}{p_t} + w_{t-1}(1-x_t)R = \\ &= f_{\omega_t}(d_{t-1}) + (w_{t-1} - p_t)R \ , \end{aligned} \tag{3.4}$$

so that the syestem reads

$$\begin{cases} p_t = \frac{\left(Rw_{t-1}+f_1+f_2\right)+\sqrt{\left(Rw_{t-1}+f_1+f_2\right)^2 - Rw_{t-1}E_{t-1}[f] - 4\ f_1 f_2}}{2\ R} \\ w_t = f_{\omega_t}(d_{t-1}) + (w_{t-1} - p_t)R \ . \end{cases} \tag{3.5}$$

Table 2: Transition probabilities between two realizations of the states of the world.

$$P(\omega_t|\omega_{t-1}) = \begin{array}{ccc} \hline & \omega_t = 1 & \omega_t = 2 \\ \hline \omega_{t-1} = 1 & \Pi_1 & 1 - \Pi_1 \\ \omega_{t-1} = 2 & \Pi_2 & 1 - \Pi_2 \\ \hline \end{array}$$

Table 3: Transition probabilities of the two equiprobable and independent states of the world.

$$P(\omega_t|\omega_{t-1}) = \begin{array}{ccc} \hline & \omega_t = 1 & \omega_t = 2 \\ \hline \omega_{t-1} = 1 & 1/2 & 1/2 \\ \omega_{t-1} = 2 & 1/2 & 1/2 \\ \hline \end{array}$$

4 Fixed Investment share

In this section we consider an agent which invests a constant fraction of wealth in the risky security.

4.1 Informed agent

We assume, in this case, that the agent operating in the market is well informed; therefore he knows how the market behaves. In other words, the probability the agent assigns to a certain realization of the state of the world equals the "real" probability. We further assume that the two state are equiprobable and independent from previous realization such that $\Pi_1 = \Pi_2 = 1/2$ (see Table 2). If we assume $R = 1$, that is the riskless interest rate equals zero, and $d_1 = f_1(d_{t-1}) = 1 - \lambda$, $d_2 = f_2(d_{t-1}) = 1 + \lambda$, then

$$E_{t-1}[f] = \frac{1}{2}(1 + \lambda) + \frac{1}{2}(1 - \lambda) = 1 \ .$$

4.1.1 Wealth Dynamics

The evolution of agent's wealth can be rewritten as

$$w_t = f_{\omega_t} + (1 - x)w_{t-1} \ , \qquad (4.1)$$

where w_t is a function of the wealth at a time $t - 1$ and of the evolving of the states of the world, and the price is simply fixed by

$$p_t = x \ w_{t-1} \ . \tag{4.2}$$

The expected value of the wealth at time t conditional on the knowledge of the wealth at time $t-1$ is

$$E_{t-1}[w_t|w_{t-1}] = 1 + (1-x)w_{t-1} \ .$$

The dynamics of the wealth over the time depends on the realizations of the state of the world, as stated in the first section, which can be shown by the transition matrix (see Table 3).

The next step consists in determining the general expression of the wealth unconditional to the instant of time the model is, and analyzes the raw and the central moment of the considered variable. The general expression for w_t reads

$$w_t = (1-x)^t \ w_0 + \sum_1^t{}_\tau \ f_{\omega_\tau}(1-x)^{t-\tau} \ , \tag{4.3}$$

and its expected value is

$$\begin{aligned} E[w_t] &= (1-x)^t \ w_0 + \sum_1^t{}_\tau \ (1-x)^{t-\tau} = \\ &= (1-x)^t \ w_0 + \frac{1-(1-x)^t}{x} \ . \end{aligned} \tag{4.4}$$

Notice that if $w_0 = 1/x$, then the expected value of wealth is stationary

$$E[w_t] = \frac{1}{x} \ . \tag{4.5}$$

In such *particular case* the wealth dynamics can be rewritten

$$w_t = \frac{1}{x} + \sum_1^t{}_\tau \ (f_{\omega_t} - 1)(1-x)^{t-\tau} \ , \tag{4.6}$$

and the second central moment reads

$$V[w_t] = E\Bigg[\Bigg(\frac{1}{x} + \sum_{1}^{t}{}_{\tau_1}\ (f_{\omega_{\tau_1}} - 1)(1-x)^{t-\tau_1} - \frac{1}{x}\Bigg)$$

$$\Bigg(\frac{1}{x} + \sum_{1}^{t}{}_{\tau_2}\ (f_{\omega_{\tau_2}} - 1)(1-x)^{t-\tau_2} - \frac{1}{x}\Bigg)$$

$$= \sum_{1}^{t}{}_{\tau_1} \sum_{1}^{t}{}_{\tau_2} (1-x)^{2t-\tau_1-\tau_2}\ E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1)]\ . \tag{4.7}$$

According to the previous assumption it is

$$E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1)] = \lambda^2\ \delta_{\tau_1,\tau_2}\ , \tag{4.8}$$

where δ is the *Kronecker delta*, that is a function of two variables which can assume two values: $\delta_{i,j} = 1$ if $i = j$ and zero otherwise. Now is it possible to solve 4.7 using the result of 4.8, obtaining

$$\begin{aligned} V[w_t] &= \sum_{1}^{t}{}_{\tau_1} \sum_{1}^{t}{}_{\tau_2} (1-x)^{2t-\tau_1-\tau_2}\ \delta_{\tau_1,\tau_2}\ \lambda^2 = \\ &= \sum_{0}^{t-1}{}_{\tau_1} (1-x)^{2\tau_1}\ \lambda^2 = \\ &= \lambda^2\ \frac{1-(1-x)^{2t}}{1-(1-x)^2}\ . \end{aligned} \tag{4.9}$$

To get a full view of the dynamics of the wealth we are now interested in analyzing how the wealth at a certain time t and the wealth at a time $t-1$ co-vary, that is in determining their correlation coefficient. We expect to find the presence of evident time dependence in the covariance function and strictly positive or negative correlation. The $cov(w_t, w_{t-1})$ is given by

$$E\big[\big(w_t - E[w_t]\big)\big(w_{t-1} - E[w_{t-1}]\big)\big] =$$

$$= E\Bigg[\Bigg(\sum_{1}^{t}{}_{\tau_1}\ (f_{\omega_{\tau_1}} - 1)(1-x)^{t-\tau_1}\Bigg)\Bigg(\sum_{1}^{t-1}{}_{\tau_2}\ (f_{\omega_{\tau_2}} - 1)(1-x)^{t-\tau_2}\Bigg)\Bigg] =$$

$$= (1-x)\ E\left[\left(\sum_{1}^{t-1}{}_{\tau_1}\ (f_{\omega_{\tau_1}}-1)(1-x)^{t-1-\tau_1}\right)^2\right]+$$

$$+E\left[(1-x)^{t-1}(f_{\omega_t}-1)\sum_{1}^{t-1}{}_{\tau_2}\ (f_{\omega_{\tau_2}}-1)(1-x)^{t-1-\tau_2}\right] = \ ,$$

but given that the realization at time t is independent of the previous realization and that

$$E[f_{\omega_t}-1]=0 \qquad \text{and} \qquad E\left[\left(\sum_{1}^{t-1}{}_{\tau_1}\ (f_{\omega_{\tau_1}}-1)(1-x)^{t-1-\tau_1}\right)^2\right] = V[w_{t-1}]\ ,$$

then

$$E\big[\big(w_t - E[w_t]\big)\big(w_{t-1}-E[w_{t-1}]\big)\big] = (1-x)\ V[w_{t-1}]\ , \tag{4.10}$$

this imply that the covariance of the distribution depends on the time, as we expected. The correlation coefficient, on the other hand, is given by

$$C = \frac{(1-x)\ V[w_{t-1}]}{\sqrt{V[w_{t-1}]\ V[w_t]}} = (1-x)\ \sqrt{\frac{V[w_{t-1}]}{V[w_t]}} = \sqrt{\frac{1-(1-x)^{2t-2}}{1-(1-x)^{2t}}}\ (1-x)\ , \tag{4.11}$$

where for $t \to +\infty$, this ratio tend to $1-x$. It turns out that the autocorrelation coefficient asymptotically tend to $1-x$, implying that each realization of wealth is positively influenced by the previous realization, and it determines in a relevant way the subsequent realization. Once determined the characteristics of the dynamic of the wealth, the following step is to analyze the influence such dynamics have on the distribution of prices.

4.1.2 Price dynamics

As assumed in the previous section, in this representative agent model the price at any time t is a function of the optimum quantity of wealth invested in the short-lived asset, and of the wealth of the representative agent at time $t-1$ (see 4.1). In order to study analytically the behavior of the variable price, let us introduce a new variable r, that we call *absolute return* or *absolute price growth.* The absolute return is simply the difference between the price at a time t and the price at a time $t-1$, in term of this model it is given by

$$
\begin{aligned}
r_{t+1} &= p_{t+1} - p_t = x(w_t - w_{t-1}) = \\
&= x\ f_{\omega_t} - x^2\ w_{t-1}\ .
\end{aligned}
\tag{4.12}
$$

The last step of the analysis consists in determining if there are any signals of predictability or correlation in price dynamics. To do so, we compute the covariance between returns, from time $t+2$ to time $t+1$. Following from 4.12, since

$$
E[r_{t+1}] = x - x^2\ \frac{1}{x} = 0\ , \tag{4.13}
$$

then

$$
\begin{aligned}
&E(r_{t+2}\ r_{t+1}) = \\
=&E[(x\ f_{\omega_{t+1}} - x^2\ w_t)(x\ f_{\omega_t} - x^2\ w_{t-1})] \\
=&x^2\ E[f_{\omega_{t+1}}]\ E[f_{\omega_t}] - x^3\ E[f_{\omega_{t+1}}]\ E[w_{t-1}] - x^3\ E[f_{\omega_t}\ w_t] + x^4\ E[w_t\ w_{t-1}] = \\
=&x^4\ E[w_t\ w_{t-1}] - x^3\left[(1+\lambda^2) + \frac{(1-x)}{x}\right] = \ ,
\end{aligned}
\tag{4.14}
$$

but from 4.10 we know that $E[w_t\ w_{t-1}] = V[w_{t-1}] + 1/x^2$, then it turns out that

$$
\begin{aligned}
E(r_{t+2}\ r_{t+1}) &= x^4\left(V[w_{t-1}] + \frac{1}{x^2}\right) - x^2 - x^3\ \lambda^2 = \\
&= x^4 V[w_{t-1}] - x^3\ \lambda^2 = \\
&= x^4\lambda^2\ \frac{1-(1-x)^{2t-2}}{1-(1-x)^2} - x^3\ \lambda^2\ ,
\end{aligned}
\tag{4.15}
$$

where for $t \to +\infty$, if $\lambda \le 1$ the covariance in absolute returns distribution asymptotically tend to zero, if $\lambda > 1$ the covariance is slightly positive for $0.5 < x < 1$, and it is slightly negative for $0 < x < 0.5$.

4.2 Uninformed agent

In the previous subsection we discussed about the dynamics of wealth and prices for the case of a representative 'informed' agent, where 'informed' means that the agent assigns the same probability to the each possible realization of the state of the world (see Table 3). Let now turn to the case of an 'uninformed' agent in which the probability the agent assigns to the realization of the states of the world depends on

Table 4: Markovian transition matrix: probabilities of realization of the states depend only on the previous realization.

$$P(\omega_t|\omega_{t-1}) = \begin{array}{c|cc} \hline & \omega_t = 1 & \omega_t = 2 \\ \hline \omega_{t-1} = 1 & \frac{1}{2} + \delta_1 & \frac{1}{2} - \delta_2 \\ \omega_{t-1} = 2 & \frac{1}{2} - \delta_1 & \frac{1}{2} + \delta_2 \\ \hline \end{array}$$

the observation of the previous realization. The aim is to analyzes the dynamics of the model and compare the results with the previous section.

4.2.1 Wealth dynamics

In this case the transition matrix can be represented by Table 4, with

$$\Pi_1 = \frac{\frac{1}{2} - \delta_2}{1 - \delta_1 - \delta_2} \qquad \text{and} \qquad \Pi_2 = \frac{\frac{1}{2} - \delta_1}{1 - \delta_1 - \delta_2} \ .$$

If we assume $\delta_1 = \delta_2 = \delta$, where δ can assume values between $-1/2$ and $1/2$ ($\delta \in (-\frac{1}{2}; \frac{1}{2})$), we obtain a symmetric transition matrix as it is shown in Table 5. Notice that

$$E[f_{\omega_t}|\omega_{t-1}] = \begin{cases} 1 + 2\lambda\delta & \text{if } \omega_{t-1} = 1 \\ 1 - 2\lambda\delta & \text{if } \omega_{t-1} = 1 \ , \end{cases}$$

but the invariant distribution of the process is $\left(-\frac{1}{2}, \frac{1}{2} \right)$ so that

$$E[f_{\omega_t}] = 1 \ .$$

As for the case of informed agent, notice that if $w_0 = 1/x$, it yields the same general expression for the wealth dynamics (4.6)

$$w_t = \frac{1}{x} + \sum_{1}^{t} {}_{\tau} \ (f_{\omega_t} - 1)(1 - x)^{t-\tau} \ ,$$

with stationary expected value of wealth (4.5).

Following the assumed procedure the next steps consist in analyzing to what extent the wealth co-varies over the time and its degree of correlation, comparing it with the result yielded by the 4.10 and by the 4.11, and in showing how such dynamics can have an influence on the working of absolute return series.

Given the transition matrix in Table 5, the probability of the two states now

Table 5: Symmetric Markovian transition matrix.

$$P(\omega_t|\omega_{t-1}) = \begin{array}{c|cc} & \omega_t = 1 & \omega_t = 2 \\ \hline \omega_{t-1} = 1 & \frac{1}{2} + \delta & \frac{1}{2} - \delta \\ \omega_{t-1} = 2 & \frac{1}{2} - \delta & \frac{1}{2} + \delta \end{array}$$

depends on the previous realization, so that for $\tau_1 = \tau_2$

$$E[(f_{\omega_\tau} - 1)^2] = \lambda^2 ,$$

for $\tau_1 = \tau_2 + 1$

$$E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1)] = \lambda^2 \left(\frac{1}{2} + \delta - \frac{1}{2} + \delta \right) = 2\lambda^2 \delta , \tag{4.16}$$

and in general

$$E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1)] = \lambda^2 \ (2\delta)^{|\tau_1 - \tau_2|} .$$

We start with the calculus of the $cov(w_t \ w_{t-1})$, that is

$$\begin{aligned} &E\big[\big(w_t - E[w_t]\big)\big(w_{t-1} - E[w_{t-1}]\big)\big] = \\ =&\sum_1^t {}_{\tau_1} \ \sum_1^{t-1} {}_{\tau_2} (1-x)^{2t-1-\tau_1-\tau_2} E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1) = \\ =&(1-x)V[w_{t-1}] + \lambda^2 \ (2\delta) \ \frac{1 - (1-x)^{t-1}(2\delta)^{t-1}}{1-(1-x)(2\delta)} \ . \end{aligned} \tag{4.17}$$

Since we defined the evolution of the states of the world as a Markov process, the second central moment of the variable wealth now reads

$$\begin{aligned} V[w_{t-1}] &= E[(w_t - E[w_t])^2] = \\ &= \sum_1^{t-1} {}_{\tau_1,\tau_2} (1-x)^{2t-\tau_1-\tau_2} \ E[(f_{\omega_{\tau_1}} - 1)(f_{\omega_{\tau_2}} - 1)] = \\ &= \sum_1^{t} {}_{\tau_1,\tau_2} (1-x)^{2t-\tau_1-\tau_2} \ (2\delta)^{\tau_1-\tau_2} \ \lambda^2 \ . \end{aligned} \tag{4.18}$$

It is possible to compute the variance of wealth with

$$V[w_t] = \sum_{0}^{t-1}{}_{i,j}(1-x)^{i+j}\ (2\delta)^{|i-j|}\ \lambda^2\ ,$$

defining V^* as the limit of variance for $t \to +\infty$

$$V^* = \lim_{t\to+\infty} V[w_t] = \sum_{-\infty}^{+\infty}{}_d\ \sum_{d}^{+\infty}{}_s(1-x)^s\ (2\delta)^{|d|}\ ,$$

with $i + j = s$ and $i - j = d$ then

$$V^* = \frac{1}{x}\left(\frac{1}{1-(1-x)(2\delta)} + \frac{(2\delta)}{1-x-2\delta}\right)\ . \tag{4.19}$$

The last step in the analysis of the wealth distribution consists in computing the covariance between w_t and w_{t-2} , that is

$$\begin{aligned}
&E\big[\big(w_t - E[w_t]\big)\big(w_{t-2} - E[w_{t-2}]\big)\big] = \\
=&\sum_{1}^{t}{}_{\tau_1}\ \sum_{1}^{t-2}{}_{\tau_2}(1-x)^{2t-1-\tau_1-\tau_2}E[(f_{\omega_{\tau_1}}-1)(f_{\omega_{\tau_2}}-1) = \\
=&(1-x)^2 V[w_{t-1}] + \lambda^2\ (1-x)\ (2\delta)\ \frac{1-(1-x)^{t-2}(2\delta)^{t-2}}{1-(1-x)(2\delta)} + \\
&+\lambda^2\ (2\delta)^2\ \frac{1-(1-x)^{t-2}(2\delta)^{t-2}}{1-(1-x)(2\delta)}\ ,
\end{aligned} \tag{4.20}$$

then

$$\lim_{t\to+\infty} cov(w_t\ w_{t-1}) = (1-x)\ V^* + \frac{2\delta\lambda^2}{1-(1-x)(2\delta)}\ , \tag{4.21}$$

that asymptotically tend to a positive value, in particular the higher is δ, the higher in magnitude is its asymptotic value, and

$$\lim_{t\to+\infty} cov(w_t\ w_{t-2}) = (1-x)^2\ V^* + \frac{2\delta(1-x)\lambda^2}{1-(1-x)(2\delta}) + \frac{(2\delta)^2\lambda^2}{1-(1-x)(2\delta)}\ . \tag{4.22}$$

4.2.2 Price dynamics

In the previous section we defined the variable ‘absolute return’ r at a certain time t as the difference between the price at a time t and the price at a time $t-1$. In order to find any signals of anomalies in price dynamics we need to analyze the covariance in returns. To do so, given that

$$E[r_{t+1}] = x - x^2 \ \frac{1}{x} = 0 \ ,$$

then it obtains

$$E(r_{t+2} \ r_{t+1}) = x^2 E[(w_{t+1} + w_t)(w_t - w_{t-1})] \ ,$$

but we know that

$$E[(w_{t+1} - w_t)(w_t - w_{t-1})] = (1-x)^{2t-1} \lambda^2 (2\delta) \ ,$$

then it turns out

$$E(r_{t+2} \ r_{t+1}) = x^2 \ (1-x)^{2t-1} \lambda^2 (2\delta) \ . \tag{4.23}$$

The covariance in absolute disstribution then is time dependent and for $t \to +\infty$ asymptotically tends to zero.

5 Simulation Experiments and Implications

In order to evaluate our model, in this section we try to extend the analytical findings of the previous sections using artificial data sets of wealth, prices and absolute returns simulated from our model, commenting the results and underlying its implications. First we fix parameter values, setting the optimal fraction of wealth invested in the risky asset to $x = 0.5$ and the shock in dividend to $\lambda = 0.5$. We set the initial level of wealth to $w_0 = 1/x$ and we simulate the model for a time horizon $T = 1000$. Similarly to the analytical study, we distinguish the simulation between the two cases: informed agent and uninformed agent.

5.1 I.i.d. Process

We start the simulation experiments with the case in which the two states are equiprobable and independent from previous realizations. First we simulate the variable wealth; its series is shown in Figure 2. In Section 4 we found the autocorrelation coefficient of wealth, arguing that it should tend to $1 - x$, implying the presence of positive autocorrelation in the series (equation 4.11). The simulation confirms our expectation, as can be seen from the figure 3 which shows the autocorrelagram and the partial autocorrelogram of wealth distribution. The PACF confirm the AR(1) nature of the process.

However, we are interested in analyzing how wealth dynamic can influence the distribution of prices and the distribution of absolute returns. We wish to find the presence of autocorrelation in returns, indicating the emergence of anomalies in returns consistent with the behavioral finance studies. Figure 4 and Figure 5 show the results of the simulation for absolute returns, in terms of time-series and autocorrelograms. These results, display the pattern we expect. As it is evident in the autocorrelogram in Figure 5, the absolute returns are negatively autocorrelated in the short-run but such correlation tend to disappear over longer horizons, pattern that confirms our analytical findings and it is also consistent with the evidence on overreaction (Fama and French 1988; Bondt and Thaler 1985). The same pattern occurs when simulating the model with constant fraction of wealth set to $x = 0.8$, as shown in Figure 6 - 9. As a consequence we have found that when the beliefs of a representative risk-averse agent on the possible realization the world equals the real probability, assuming states equiprobable and independent, even if we do not assume any kind of biases in the representative agents beliefs, the risk-aversion of the agent, alone, can determine the emergence of anomalies in the financial market.

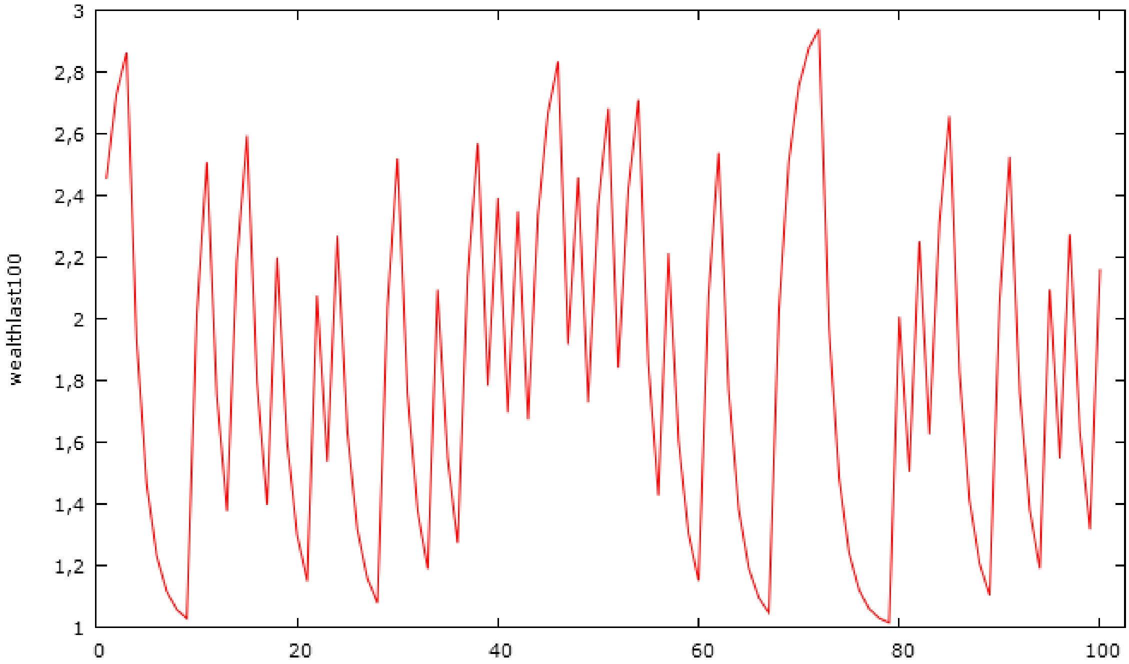


Figure 2: Wealth time-series for the last 100 periods, $x = 0.5$, $\lambda = 0.5$

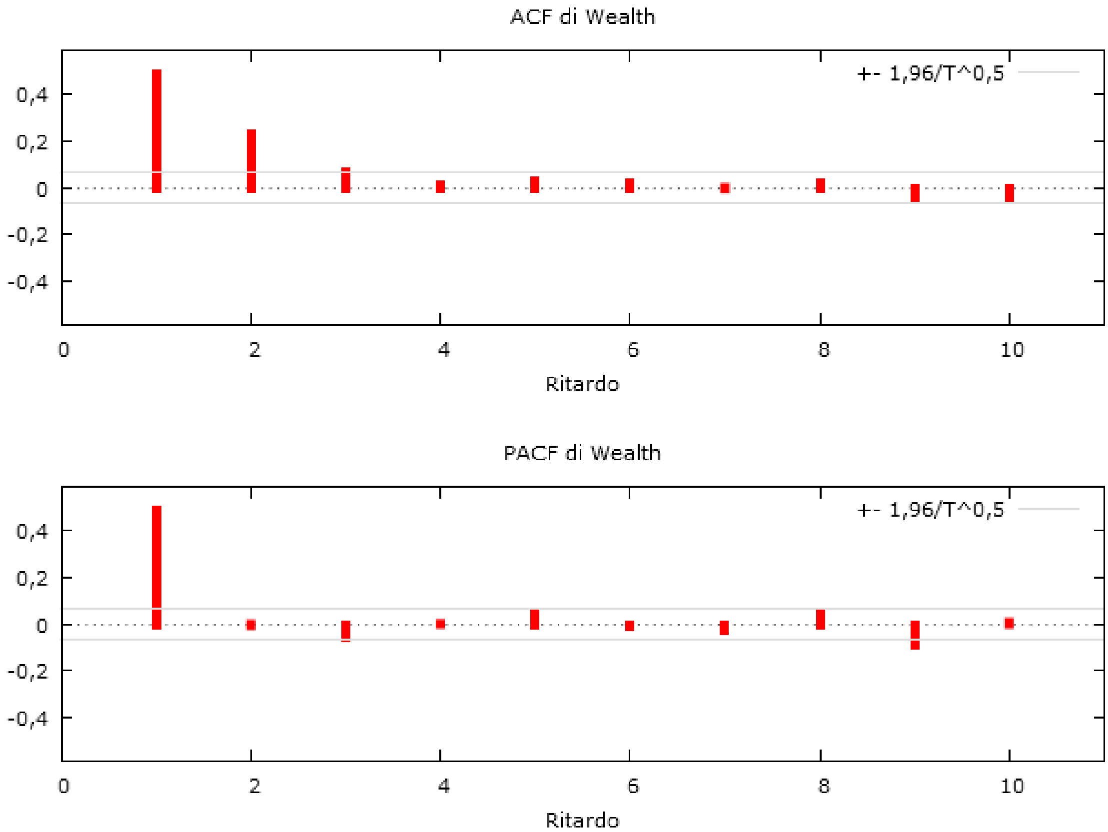


Figure 3: Wealth autocorrelogram $lag = 10$

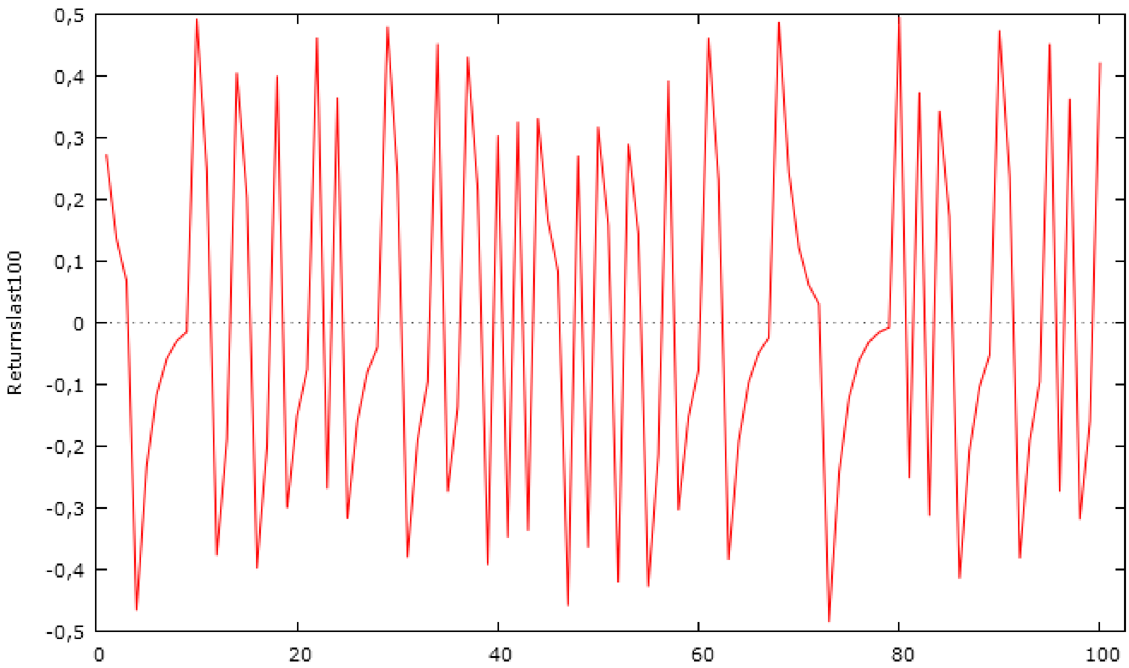


Figure 4: Returns series for the last 100 periods, $x = 0.5$, $\lambda = 0.5$

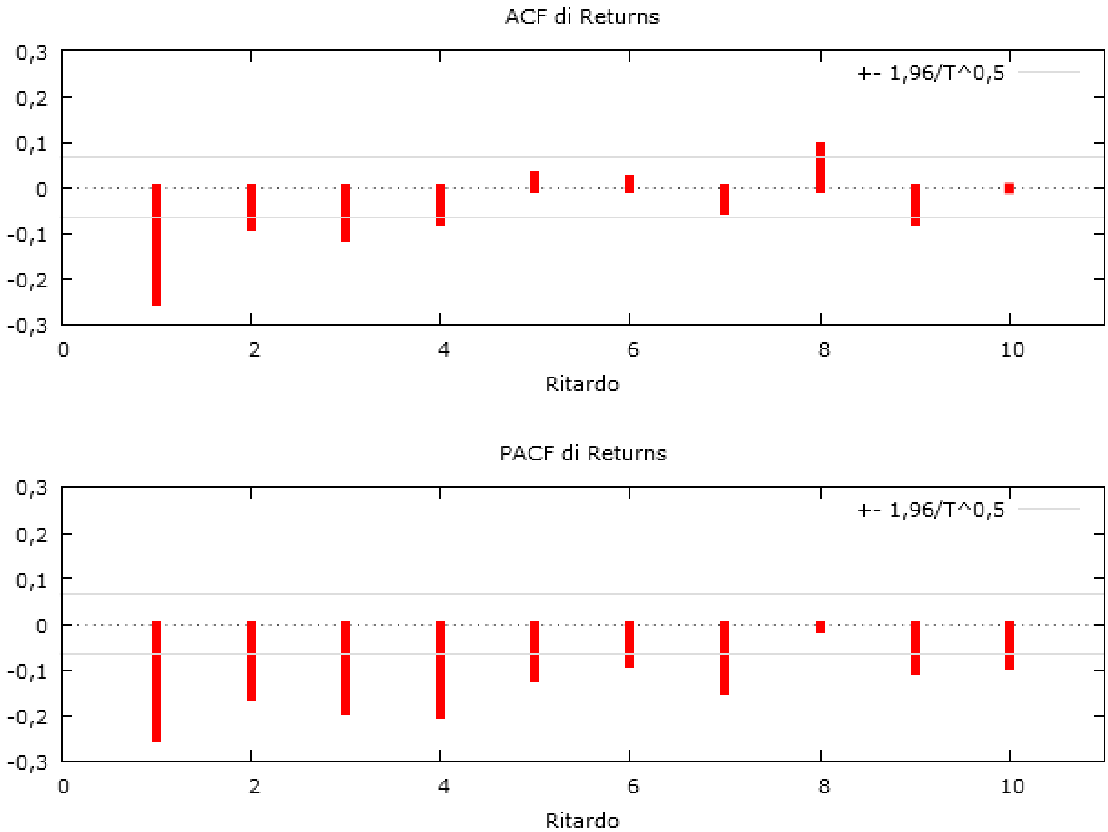


Figure 5: Returns autocorrelogram $lag = 10$

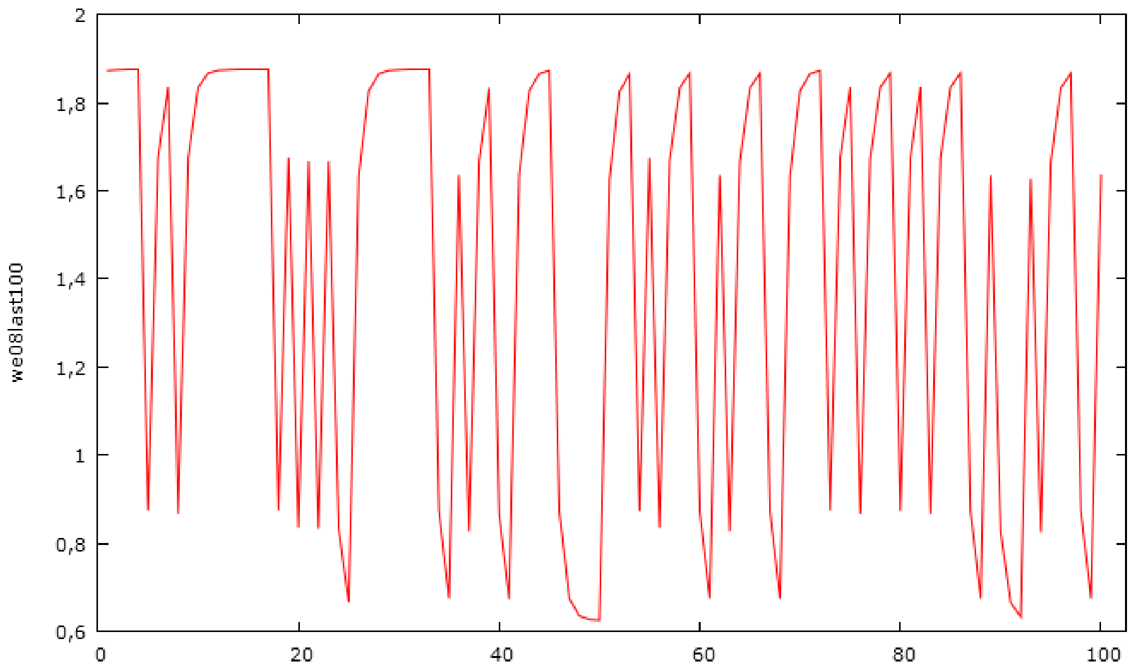


Figure 6: Wealth series sfor the last 100 periods, $x = 0.8$, $\lambda = 0.5$

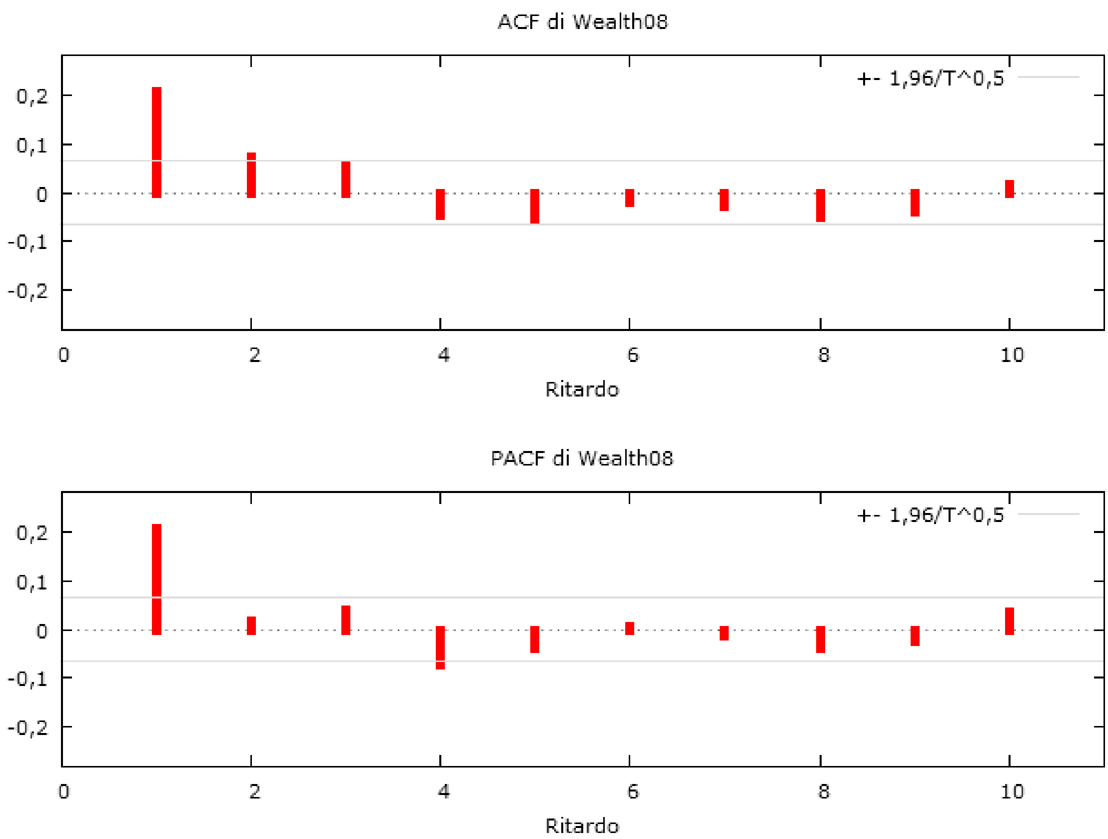


Figure 7: Wealth autocorrelogram $lag = 10$, $x = 0.8$

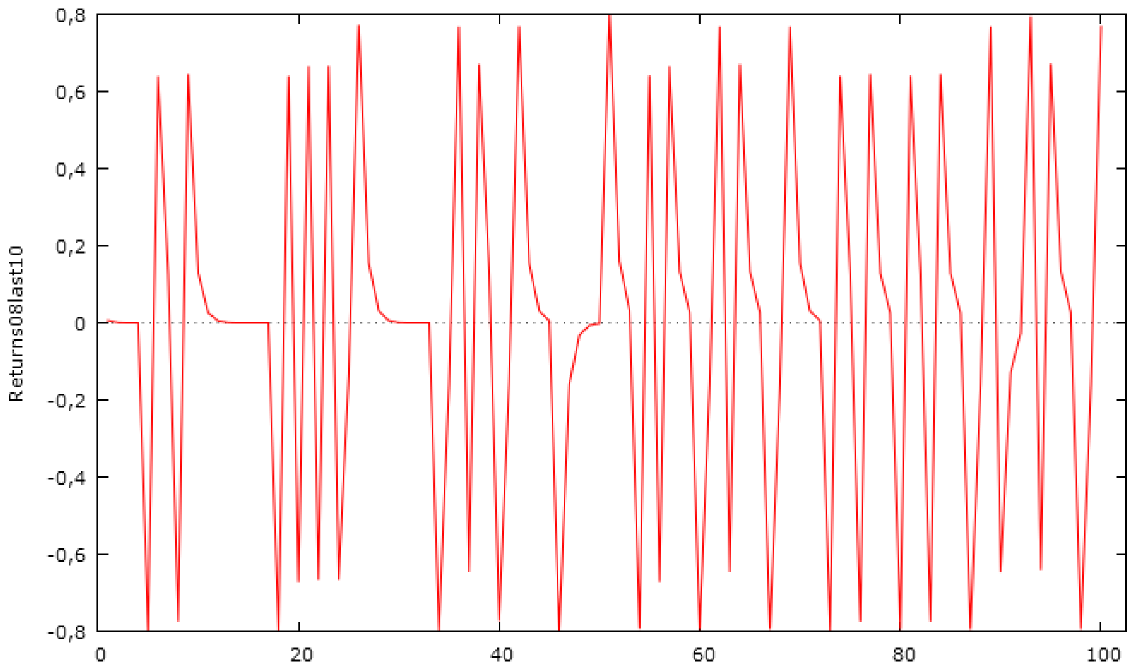


Figure 8: Returns series for the last 100 periods, $x = 0.8$, $\lambda = 0.5$

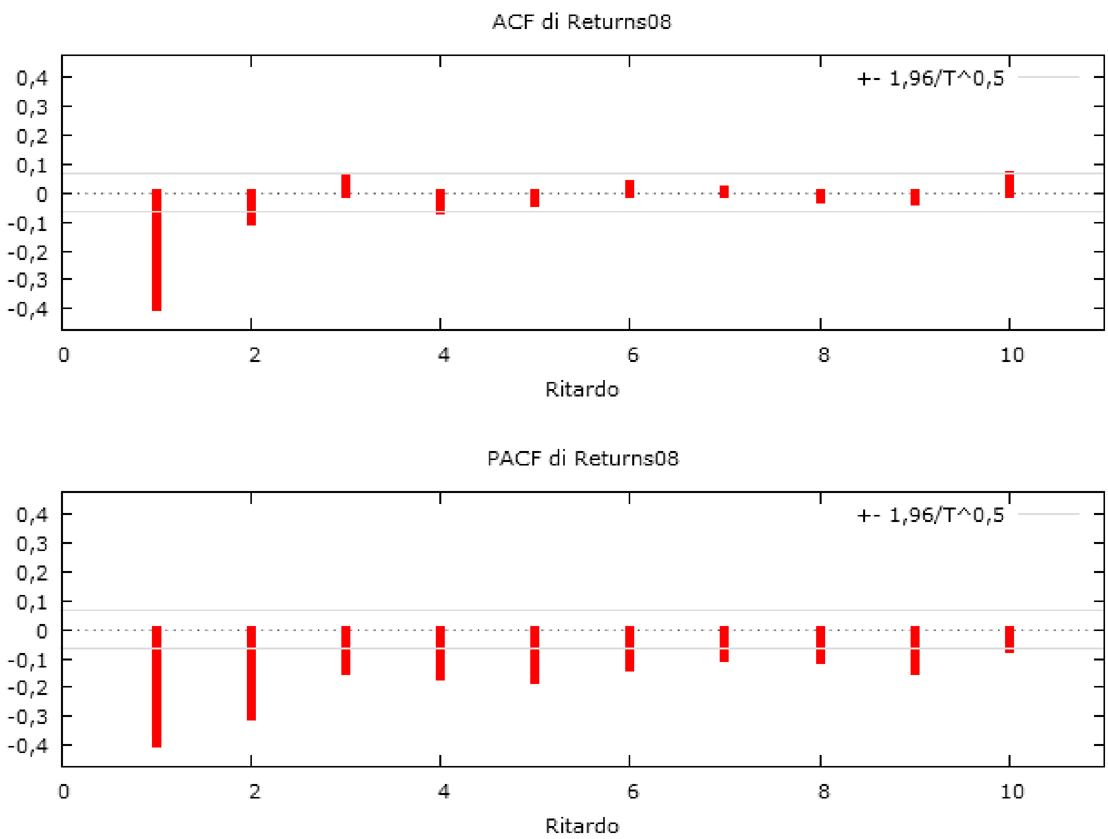


Figure 9: Returns autocorrelogram $lag = 10$, $x = 0.8$

5.2 Markovian process

In the case that we called uninformed we assumed that the realization of the state the world at a time t depends on the realization observed previously, at a time $t-1$. In particular we studied analytically the case of symmetric transition probabilities: that is, observing a certain realization, for example assume it is positive, if δ is positive it is more likely that the subsequent realization will be positive as well, on the other hand, it is more likely to revert if δ is negative.We start the experiment simulating the time series of wealth, prices and related absolute return in the case of positive δ, in particular we set $\delta = 0.3$. Looking at Figure 11, which shows the autocorrelagram and the partial autocorrelogram, it is evident that wealth distribution is positively autocorrelated, confirming our analytical findings.

However, as for the informed case, the main aim is to examine the dynamics of the absolute return. In the case of $\delta > 0$ we the autocorrelogram of returns shown in Figure 13 confirm our expectations, returns are positively correlated in very short-run implying that the sign of return could be a good predictor for the subsequent return, but in the short-run tend to be negatively autocorrelated. Such results are consistent with several empirical studies (Cutler et al. 1991), which found that such positive autocorrelation exsists only in the very short run.

Let now turn to the case of $\delta < 0$. In particular we set $\delta = -0.3$ and then we run the simulation. Both the wealth distribution and the price distribution follow the pattern we derived in the analytical study (Section 4.2). In fact, as shown in Figure 15, wealth distributions is positively autocorrelated. For what concern the evolution of returns series, Figure 17 shows that returns are negatively autocorrelated in the short-run and then such autocorrelation tend to disappear over longer horizons. This result confirms our analytical findings, and it is also consistent with a mean-reverting behavior of returns treated in the literature (Poterba and Summers 1988; Lakonishok et al. 1994). As a consequence we have shown that in a representative agent framework in which the evolution of the states of the world follows a Markov process, the risk-aversion of the agent that operates in the market, can cause the emergence of signals of inefficiency in the stock market.

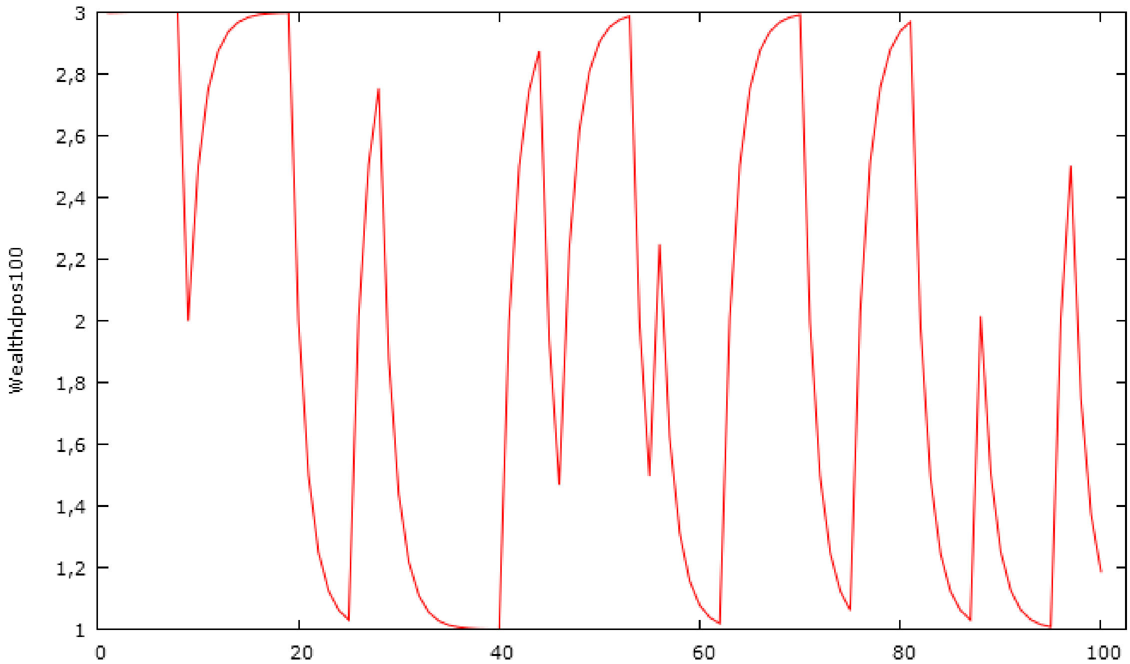


Figure 10: Wealth time-series for the last 100 periods, $x = 0.5$, $\lambda = 0.5$, $\delta = 0.3$

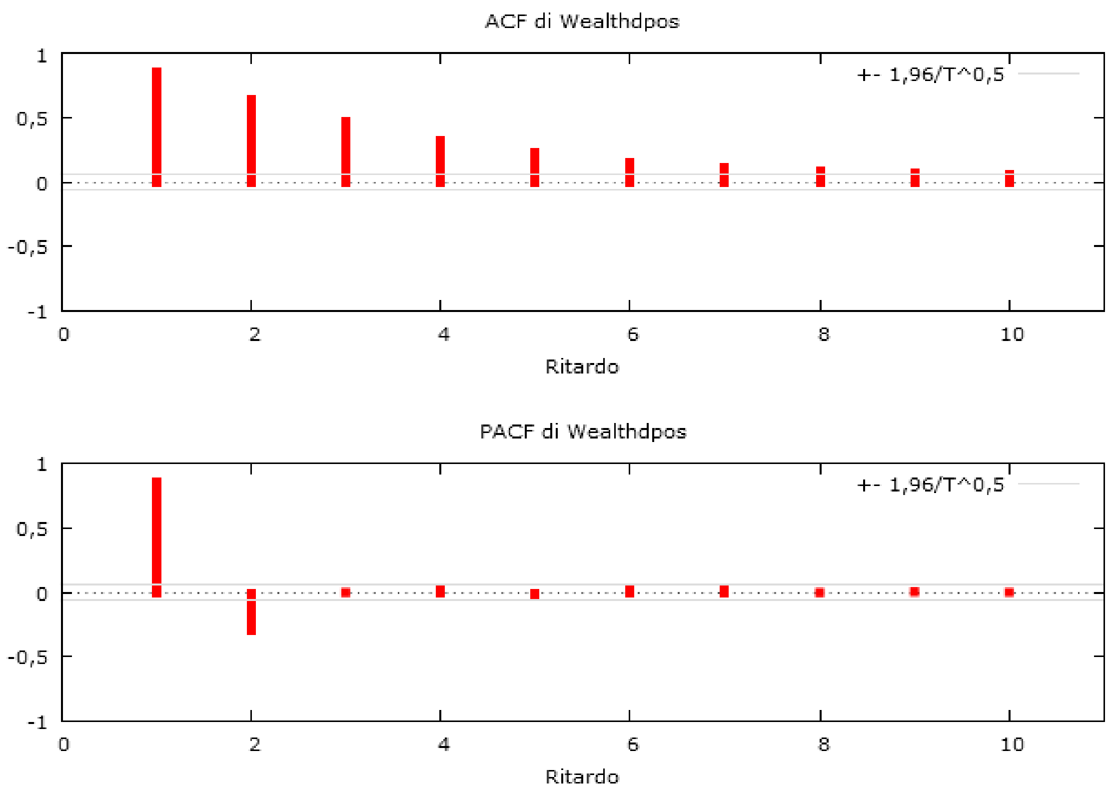


Figure 11: Wealth autocorrelogram $lag = 10$, $\delta = 0.3$

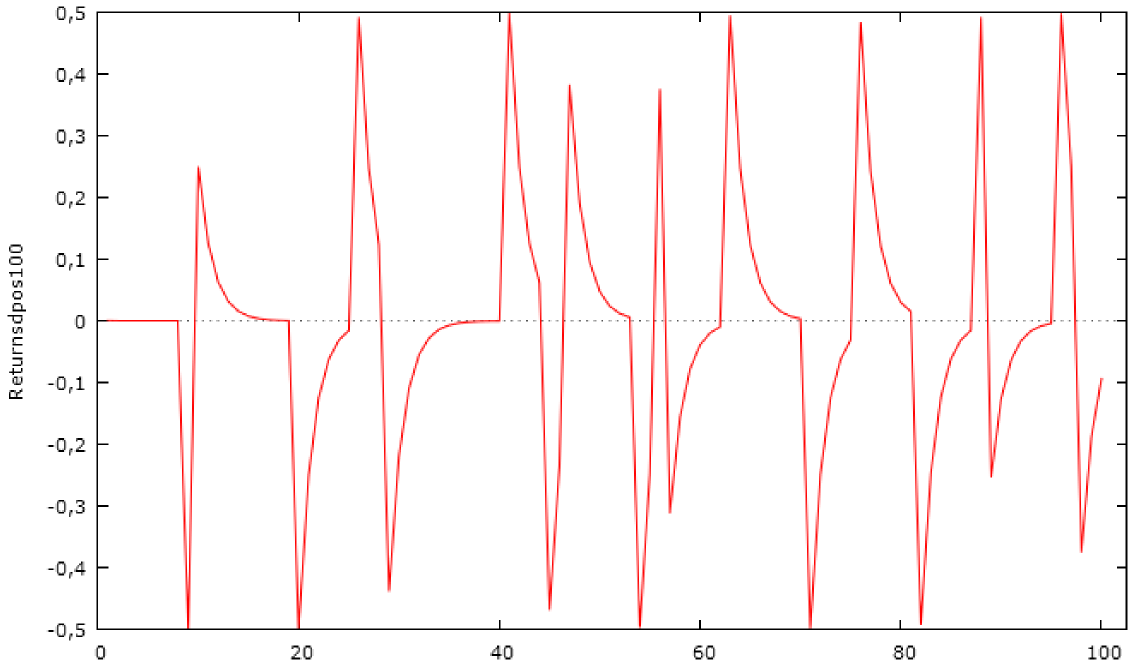


Figure 12: Returns series for the last 100 periods, $x = 0.5$, $\lambda = 0.5$, $\delta = 0.3$

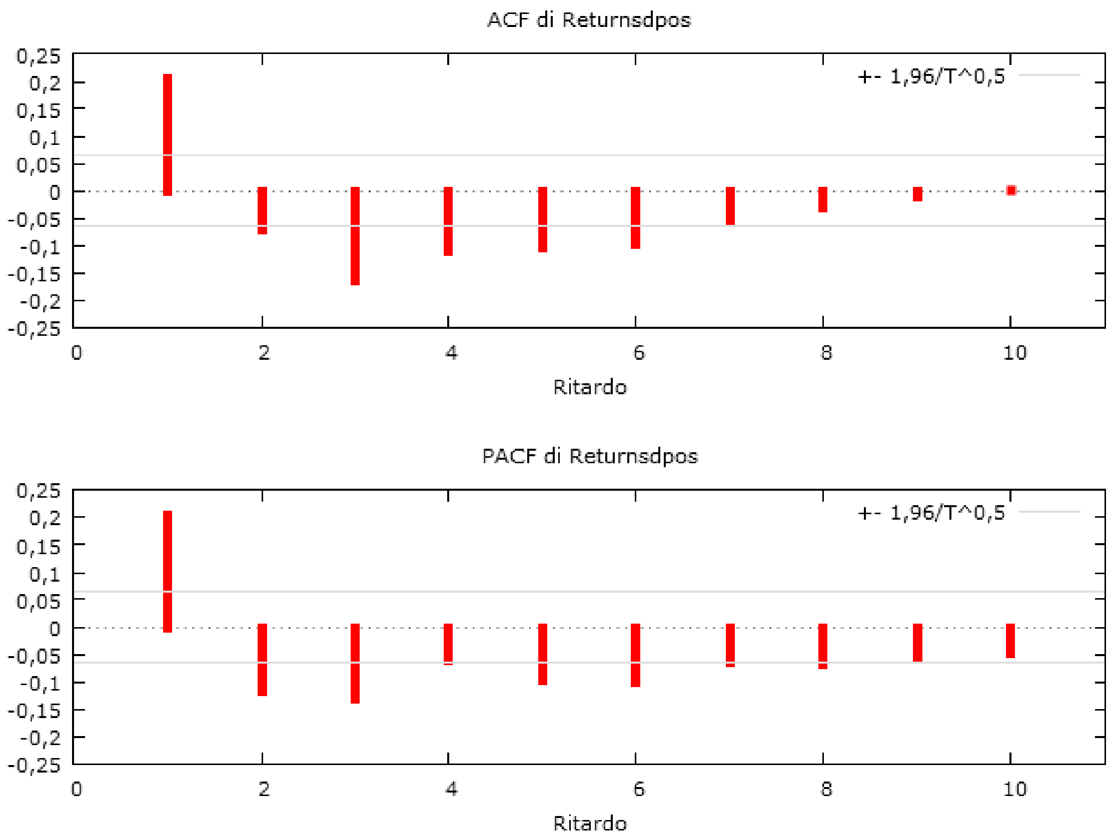


Figure 13: Returns autocorrelogram $lag = 10$, $\delta = 0.3$

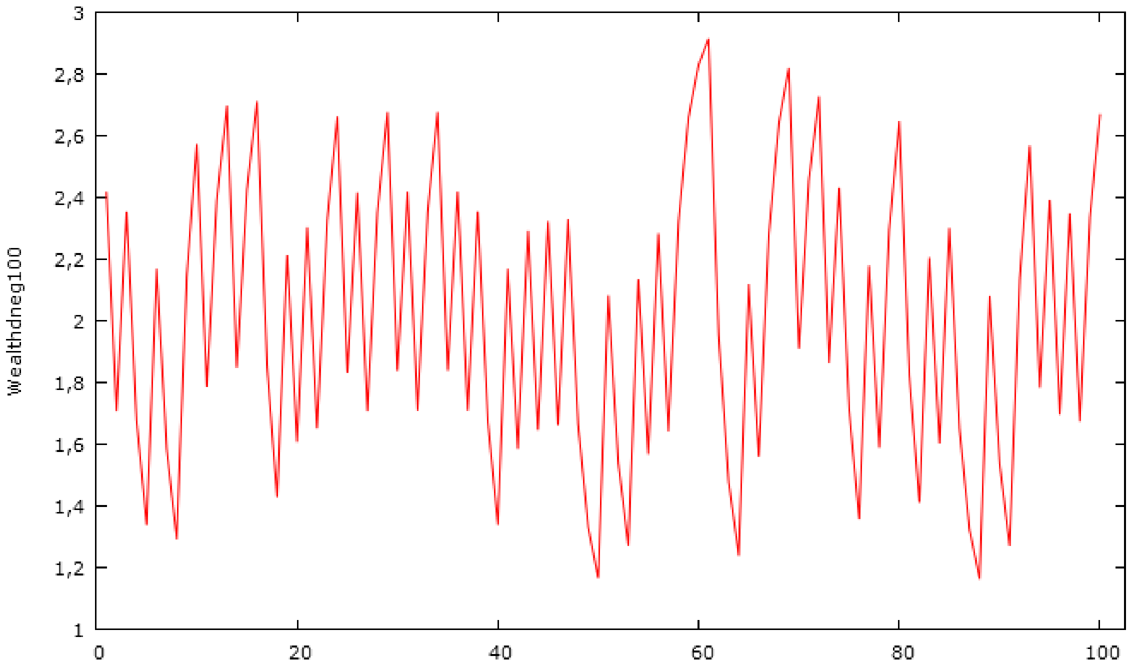
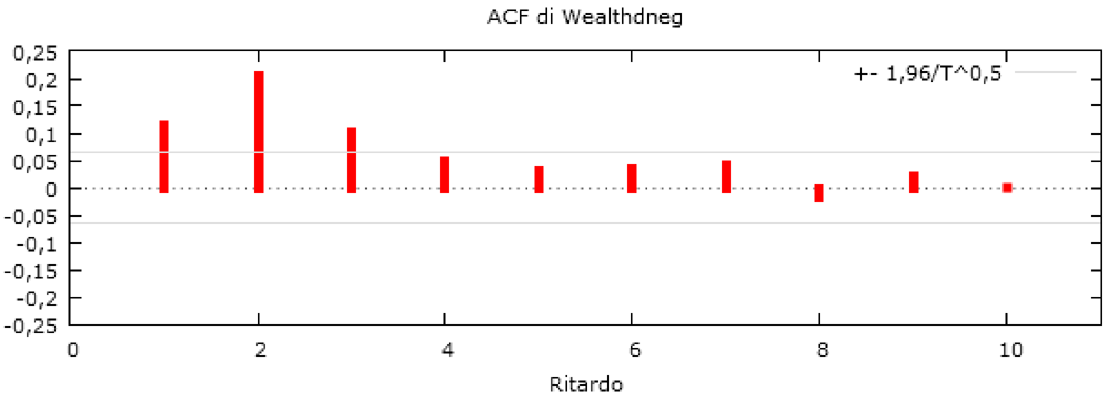


Figure 14: Wealth series for the last 100 periods, $x = 0.8$, $\lambda = 0.5$, $\delta = -0.3$



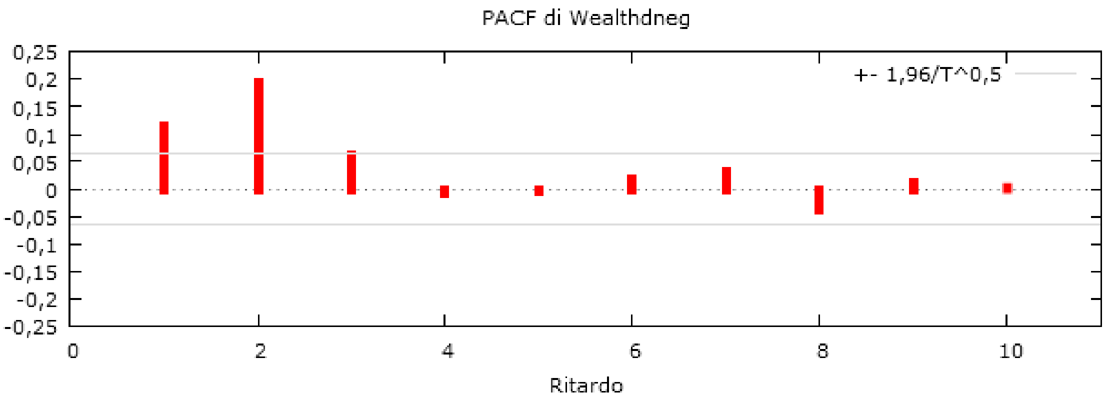


Figure 15: Wealth autocorrelogram $lag = 10$, $\delta = -0.3$

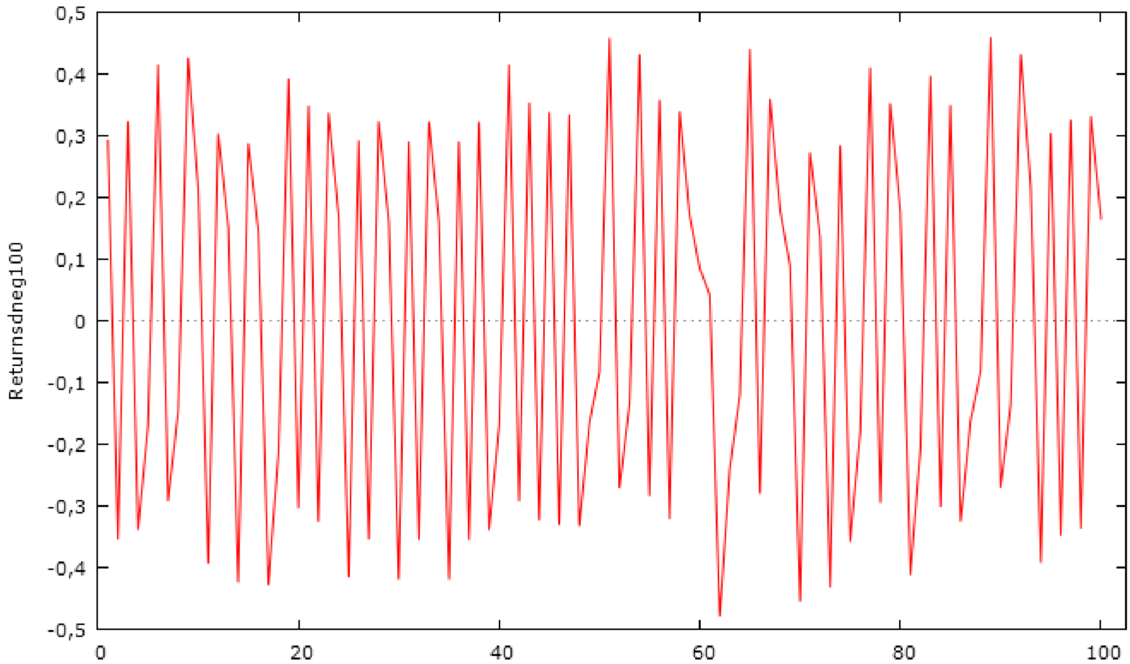


Figure 16: Returns series for the last 100 periods, $x = 0.5$, $\lambda = 0.5$, $\delta = -0.3$

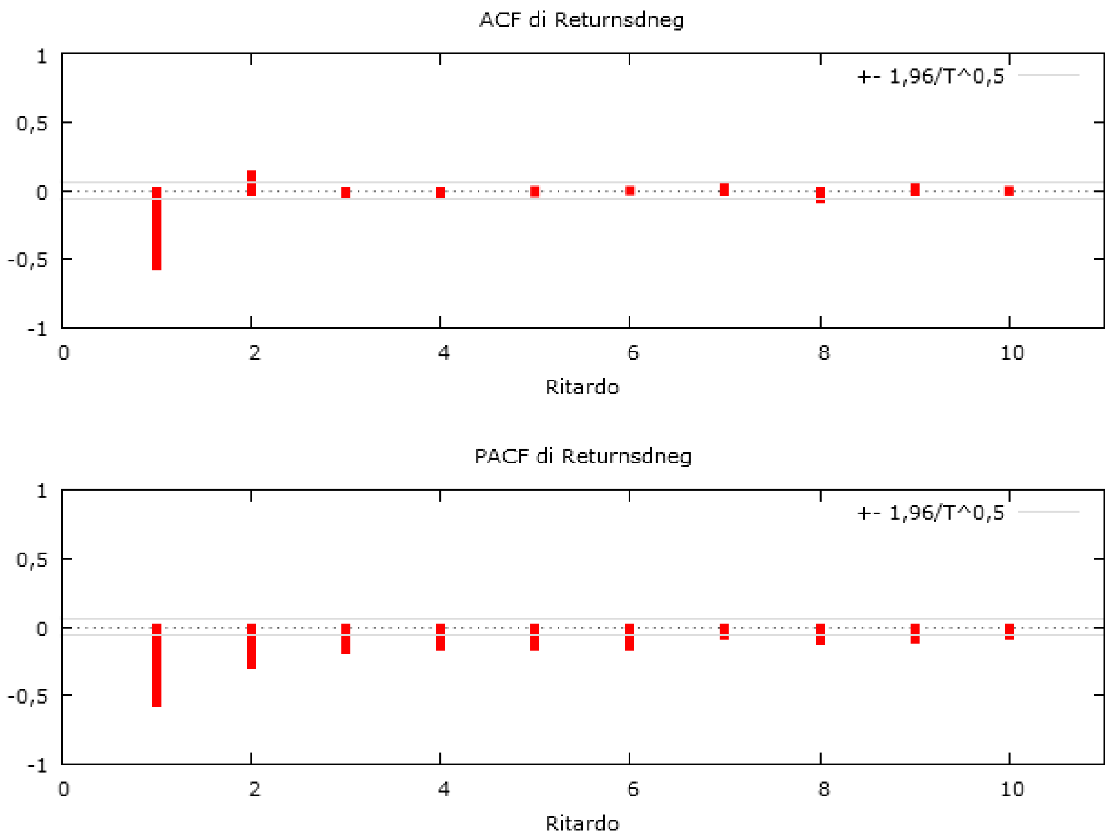


Figure 17: Returns autocorrelogram $lag = 10$, $\delta = -0.3$

6 Conclusions

Behavioral finance has provided both theory and evidence which suggest what deviations of securities from fundamental values are likely to be, and why the can persist over long time without being eliminated. Although empirical evidence was the real first instrument of behavioralists against efficient market theorists, the debates over market efficiency continue, so that behavioral finance derive its strength by its behavioral explanations of the so-called anomalies, which materialized in theories of investor behavior. The usefulness of behavioral finance lies in offering a richer description of investor behavior than those captured by fully rational utility maximizers with limited heterogeneityfor example in risk preferences by giving a collection of possible heuristics and biases, which have been documented in a financial, or sometimes a more general, decision-making setting.

In this context take place several behavioral models that have been developed during the last 20 years, which try to explain, on the theoretical ground, the emergence of market anomalies, return predictability, underreaction and overreaction, momentum strategy and others. Despite many behavioral models, our model does not take into account particular biases that can affect the investors behaviors, such as representativeness and conservatism in Barberis et al. (1998) or overconfidence and self-attribution in Daniel et al. (1998), rather it wonders if and to what extent, a risk-averse agent can be considered has biased in its beliefs and then determine stock price deviations or returns autocorrelation. In our model there is a representative risk-averse agent, which behaves as expected utility maximizers and invests a constant fraction of wealth in a short-lived risky asset. Further we assumed that the world can be in two states, so that we studied two different cases: one in which the two states are equiprobable and independent from previous realization and one in which the realization of the states depend only on the previous realization. After deriving analytically the dynamics of wealth, price and returns, we simulated the model. The results of the simulation showed that in both cases returns are autocorrelated, implying that returns are not completely unpredictable as stated by market efficiency, even when agent has not biased beliefs.

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