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General disequilibrium with log-linear prices

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Abstract

This paper presents a framework for estimating non-Walrasian models with many markets based on the virtual price approach in Lee (1986). The paper discusses an open economy multi-market non-Walrasian model with many agents and government production. The modeling of the labor markets is built on the assumption that each combination of worker and firm is a separate micro labor market. The econometric specification in the paper assumes log-linear virtual prices. Despite the use of such a simple specification it is apparent that when there are a large number of markets, the computational burden of estimation becomes heavy due to the large number of possible rationing regimes. The model presented in the paper can be viewed as a basis for either doing econometric work within a multi-market representative agent framework or for developing methods for aggregating across micro markets.

Keywords: disequilibrium, virtual prices, econometric framework JEL: C1 C62 D5

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1 Introduction

This paper discusses a multi-market non-Walrasian model with many agents which can be used for empirical work when there are a large number of markets. The framework discussed in the following is mainly an extension of the virtual price approach suggested by Lee (1986). It takes into account that there are many agents in the economy and includes an open economy and government production. It is assumed that exports, investment, and the budget constraints of the government

firms are exogenous in the model. Imports, the trade surplus, tax revenue, the public budget deficit, and changes in the money supply are endogenous.

There are two main conclusions which can be drawn from the paper. One is a restatement of the conclusion in Lee (1986) that it is possible to find computationally tractable expressions for multimarket non-Walrasian models using a virtual price approach. The second conclusion is that the assumption that each combination of firm and consumer is a separate micro labor market leads to a simplification of the modeling of these markets. Even so, it is clear that estimation in the presence of a large number of markets is still very cumbersome due to the large number of possible regimes which must be handled. It therefore seems that in econometric work one is either forced to working with representative agent models such as the example discussed at the end of the paper or econometric work must be based on some type of aggregation approach.

During the thirty years that have gone since the seminal work of Barro and Grossman (1971) appeared, there has been a steady stream of theoretical and empirical work concerning *non-Walrasian* models of the economy. It has been argued that such models are important because it has been observed that quantities often adjust faster than prices and because a non-Walrasian framework can be viewed as a generalization of the traditional Walrasian framework. Most empirical work has been based on *fixed-price models* where prices are assumed fixed in the short run without any explicit modeling (or very ad hoc modeling) of price processes.

Econometric work on non-Walrasian models has mainly been based on models with only one or two markets, such as the canonical neo-Keynesian model first introduced by Barro and Grossman (1971). This model is based on a very stringent interpretation of the economy, where the economy suddenly shifts from one regime to another. The *smoothing by aggregation approach* first suggested by Muellbauer (1978) and used in many studies, such as Sneessens and Drèze (1986), Lambert (1988), and Drèze and Bean (1990), give a more flexible interpretation of the canonical two-market model, but is still embedded within a two-market framework. Being confined to a two-market setup limits the possible empirical uses of the non-Walrasian approach. A more general framework would allow us to study empirically models with more than two markets such as the open economy models presented in Neary (1980) and Cuddington, Johansson, and Lõfgren (1984) or to study the interaction between different parts of the labor market split up by production sector and worker qualifications.

Lee (1986) shows that a *virtual price approach* makes it computationally possible to estimate econometric models with a large number of markets. In his paper Lee considers the situation with two representative agents and many markets. His approach relies on separable utility and production functions and thereby on a very simplified modeling of spillovers. The specification of *spillovers* says how rationing in one market will influence behavior in other markets. Lee's

paper shows that a fixed-price description using virtual prices is equivalent to the fixed-price specification inherent in both the Ito (1980) and the Gourieroux, Laffont, and Monfort (1980) spillover specifications, even though the excess demand and supply functions will be different.

The model which is presented in the following is based on the assumption that prices and wages do not instantaneously clear markets. Prices and wages may be flexible over time. Our assumption only excludes the case where prices continuously clear the markets. The model therefore applies both if the economy is characterized by price and wage rigidities as argued by Romer (1993) or by market failure as argued by Greenwald and Stiglitz (1993). We only assume that at any given moment in time the economy is not necessarily in a Walrasian equilibrium. We assume instead that it is in a Drèze equilibrium. A *Drèze equilibrium* is a set of transactions that are such that they are the result of utility and profit maximization subject to all quantity constraints that exist, that only one side of each market can be rationed at a given time, and that net transactions of each non-tradeable good sums to zero across the economy as a whole. This definition differs from that normally used in that it includes the possibility of an open economy.

The modeling of the labor markets is built on the assumption that each combination of worker and firm is a separate labor market, an approach that has strong similarities with that of Benassy (1987). The main difference is that price taking behavior is assumed in the following instead of the monopolistic competition assumption of Benassy's paper.

It is important to note that this paper is concerned with developing a framework for analyzing a given set of observed transactions. The observed transactions are viewed as being the result of a unobserved rationing mechanism. Rationing in the economy is revealed implicitly through the difference between the observed transactions and the transactions that are optimal for the agents. Specifically this will, in the virtual price approach, be reflected in the differences between virtual prices and market prices. In the following a general non-Walrasian model is presented first and subsequently we derive the inverse demand and supply functions associated with this model. These inverse relationships depend on market prices and observed transacted quantities.

2 The model

Consider a simple general equilibrium model with four types of agents: $\$M_1$ private firms competing in world markets indexed by $j = 1, \cdots M_1$, $M_2 - M_1$ private firms sheltered from international competition indexed by $j = M_1 + 1, \cdots, M_2$, $M - M_2$ government firms indexed by $j = M_2 + 1, \cdots, M$, and N consumers/workers indexed by $j = M + 1, \cdots, M + N$. Consumers maximize utility, private firms maximize profits constrained by their revenues in the previous period while the government firms maximize profits given an exogenously (politically) set budget constraint. All

agents take prices as given in both input and output markets. Government production is included because it constitutes a significant portion of the output of many economies. Such production is often supplied to the public at prices that are so low that they don't clear the markets. A typical example of such production is public health care. It is natural to treat the output from government production as non-traded goods consisting mostly of services. Government production is assumed to be used only by consumers, consists only of individual products (we assume there are no public goods), is not an input to other production, is not an investment good, and is never exported. The government sector may buy investment goods from private firms and from abroad.

Each firm uses money, labor and output from other firms as inputs and produces one good, so that there are M produced goods in the model including imported goods. Imports are the difference between transacted and produced quantities of the goods $j = 1, \cdots M_1$. If there are imports of a good not produced nationally, we assume that there is a hypothetical firm which could have produced the good but which finds such production unprofitable. We assume a non-symmetry between imports and exports. Exports are governed by longer term contracts and are exogenous, while imports are residually determined and endogenous. Consumers are never rationed in their demand for goods that can be imported, but firms may be rationed in their supply of these same goods because they cannot negotiate new export contracts quickly enough. There is one non-produced commodity in the model which will be referred to as money. Money enters both the utility and production functions as a means to facilitate transactions and because it is the sole means of transferring liquidity over time (there are no financial markets in the model other than money).

As mentioned earlier, each combination of firm and consumer is considered a separate labor market. Consumers can supply labor to all firms and each firm supplies goods to all consumers. In principle it is possible for a firm to use all types of labor and all types of commodities as inputs and it is possible for an individual to be employed in all the firms in the economy and to consume all types of commodities. Such a large and general opportunity set both for the firm and especially for the individual will naturally lead to a large number of corner solutions which it is necessary to take into consideration. There is of course nothing in the above formulation which precludes many types of labor or commodities being the same and being exchanged in the same market at the same price. The above leads to there being M product markets, $M \cdot N$ potential labor markets consisting of the M firms times the N consumers, and a market for money.

We assume that the length of production for all firms is one period, implying that for all inputs that are chosen in the current period, output and the resulting revenues will first accrue in the next period. This results in the private firms being constrained by a budget constraint similar to

that of the consumer[1]. This approach is similar to the one taken in for example Bõhm and Lèvine (1979). They argue that the firm's maximize a complex intertemporal function, implying that one might just as well represent the firm's criterion function by a general utility function as by a profit function. In addition the firms face intertemporal financing constraints, implying that they face budget constraints similar to the consumer.

Production and consumption inputs of a good are transacted on the same product market at the same price. In the following we only look at short term equilibria and consider the capital stock, investment, exports, and government behavior (tax rates and the budget constraint of the government firms) as exogenous, along with prices and wages. The main reason for considering these as exogenous is analytical tractability, but it can be argued that decisions regarding these variables cover a longer time period than decisions regarding consumption and production. This argument can be seen as an extension of the usual reasoning behind fixed-price models, that quantities adjust faster than prices. We assume that the model has a fixed-price equilibria of the Drèze type and do not consider the dynamics which can occur over time. We assume that the demand for money is always satisfied.

Tables 1, 2, 3, and 4 show the main variables in the model. The variable x_{ji} denotes the use of good j either for consumption or as a production input by agent i. The supply of labor from individual i to firm i is denoted l_{ij} and the maximum number of hours which it is possible for an individual to work is denoted $\overline{L}$. Imports of the products that can be produced by the M_1\$M_{1}\$ firms facing foreign competition are denoted imp_j. For notational convenience we let $\boldsymbol{x}_i$ be a vector of the goods used by agent i and $\boldsymbol{l}_i$ be a vector of the labor supplied or demanded by agent i, where we have

$$\mathbf{x}_i = \begin{cases} [x_{1i}, \cdots, x_{i-1\,i}, x_{i+1\,i}, \cdots, x_{M_2 i}] & \text{when } j = 1,\ldots,M, \\ [x_{1i}, \cdots, x_{Mi}] & \text{when } j = M{+}1,\ldots,M{+}N, \end{cases}$$

$$\mathbf{l}_i = \begin{cases} [l_{M+1\,i}, \cdots, l_{M+N\,i}] & \text{when } j = 1,\ldots,M, \\ [l_{i1}, \cdots, l_{iM}] & \text{when } j = M{+}1,\ldots,M{+}N. \end{cases}$$

The variable inv_{jk} denotes the amount of good j used by firm k to increase it's capital equipment, $y_{inv,j}$ denotes domestic production of good j for use as capital, and $y_{\exp,j}$ denotes the demand for exports delivered from firm j. These are exogenous variables. Because of imports, it is possible for $\sum_k inv_{jk} > y_{inv,j}$ for $j = 1, \cdots, M_1$. Production for consumption and for use as production inputs is assumed to be endogenous and is denoted y_j. The exogenous investments made by firm k, inv_{jk}, are necessarily related to the production of investments goods by each firm, but since both investments and production of investments goods are exogenous in the model such

[1] We would get similar results in the following if we assumed that the firm was not constrained in this manner. The assumption seems plausible and facilitates proving the existence of a Drèze equilibrium (see appendix A).

Table 1: Use of goods and services in the model

	used by firms			*used by consumers*			*used for invest. + exports*
	1	...	M	$M+1$	...	$M+N$	
money holdings							
	m_1	...	m_M	m_{M+1}	...	m_{M+N}	
firms facing foreign competition							
product 1	-	...	x_{1M}	$x_{1\,M+1}$	...	$x_{1\,M+N}$	$\sum_k inv_{1k} + y_{exp,1}$
product 2	x_{21}	...	x_{2M}	$x_{2\,M+1}$	...	x_{2M+N}	$\sum_k inv_{2k} + y_{exp,2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
prod. M_1	$x_{M_1\,1}$	...	$x_{M_1\,M}$	$x_{M_1\,M+1}$	...	$x_{M_1\,M+N}$	$\sum_k inv_{M_1\,k} + y_{exp,M_1}$
firms sheltered from foreign comp.							
prod. M_1+1	$x_{M_1+1\,1}$	...	$x_{M_1+1\,M}$	$x_{M_1+1\,M+1}$	...	$x_{M_1+1\,M+N}$	$\sum_k inv_{M_1+1\,k}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
prod. M_2	$x_{M_2\,1}$	...	$x_{M_2\,M}$	$x_{M_2\,M+1}$	...	$x_{M_2\,M+N}$	$\sum_k inv_{M_2\,k}$
government firms							
prod. M_2+1				$x_{M_2+1\,M+1}$	...	$x_{M_2+1\,M+N}$	
$\vdots$				$\vdots$	$\ddots$	$\vdots$	
product M				$x_{M\,M+1}$	...	$x_{M\,M+N}$	
labor							
labor M+1	$l_{M+1\,1}$	...	$l_{M+1\,M}$				
$\vdots$	$\vdots$	$\ddots$	$\vdots$				
labor M+N	$l_{M+N\,1}$	...	$l_{M+N\,M}$				

Table 2: List of variables

	Production and profits
imp_j :	imports of product j in competition with firm j's production
$y_{inv,j}$:	firm j's production of goods used as capital investment in other firms
$y_{exp,j}$:	firm j's production of goods for export
Y_j :	production for investment and export ($Y_j = y_{inv,j} + y_{exp,j}$)
y_j :	total production in firm j minus production for inv. and export
π_j :	profits in firm j from the present period
π_j° :	profits in firm j in the preceding period
	Prices
v_{1j} :	price of good j when used as capital investment
v_{2j} :	price of exports delivered by firm j
p_j :	after-tax price of good j for other uses
w_{ij} :	price of labor supplied by consumer i to firm j
	Stocks
K_j° :	the stock of capital in firm j at the beginning of the period
C_j° :	agent j's total stock of money at the beginning of the period
m_{fj}° :	money holdings by firm j at the beginning of the period
m_{ci}° :	money holdings by consumer i at the beginning of the period
$\bar{m}^\circ$:	aggregate money supply at the beginning of the period
$\bar{m}$:	aggregate money supply at the end of the period
$\Delta\bar{m}$:	changes in aggregate money supply ($\bar{m} - \bar{m}^\circ$)
	Taxes
t_{1i} :	average tax rate on agent i's labor income
t_{2j} :	average tax rate on commodity j
t_{3j}° :	lump-sum tax transfer to agent i at the beginning of the period
T :	total tax receipts

Table 3: List of variables continued

Constraints

$\overline{x}_{ji}$:	upper bound on agent i's net purchase of good j
$\bar{l}_{ij}$:	upper bound on the net purchase of labor
$\underline{l}_{ij}$:	lower bound on the net purchase of labor
$\underline{y}_j$:	upper bound on the net sale of good j by firm j
$\bar{L}$:	maximum number of hours which it is possible to work

Drèze demands and supplies

y_j^s :	firm j's Drèze supply of good j as a consumption or production input
x_{ji}^d :	agent i's Drèze demand for good j as a consumption or production input
l_{ij}^s :	consumer i's Drèze supply of labor to firm j
l_{ij}^d :	firm j's Drèze demand for consumer i's labor ($l_{ij}^d = l_{ij}^s = l_{ij}^*$ in equilibr.)
m_{fj}^d :	firm j's Drèze demand for money
m_{ci}^d :	consumer i's Drèze demand for money

Vectors

p^* :	vector of prices $p_1, \ldots, p_M$
$w_{i\cdot}$:	vector of wages $w_{i1}, \ldots, w_{iM}$ faced by consumer i
$w_{\cdot j}$:	vector of wages $w_{M+1\,j}, \ldots, w_{M+N\,j}$ faced by firm j
$\overline{x}_{\cdot j}$:	vector of upper constraints x_{kj} faced by agent j in the goods market
$\bar{l}_{\cdot j}$:	vector of upper constraints l_{ij} faced by firm j in the labor market
$\underline{l}_{i\cdot}$:	vector of lower constraints l_{ij} faced by consumer i in the labor market
$\mathbf{x}_i$:	vector of the goods used by agent i
$\mathbf{l}_i$:	vector of the labor supplied or demanded by agent i
$\mathbf{x}_i^d$:	vector of agent i's Drèze demands for goods
$\mathbf{l}_i^d$:	vector of firm i's Drèze demand for labor
$\mathbf{l}_i^s$:	vector of consumer i's Drèze supply of labor
$\mathbf{m}^d$:	vector of all the agents' demands for money

Table 4: List of variables continued

	Vectors continued
$\mathbf{l}^*$:	vector of all transacted quantities of labor
$\mathbf{x}^d$:	vector of all Drèze demands for goods
	Virtual prices
ξ^l_{ij} :	consumer i's virtual price for delivering labor to firm j
ξ^x_{ji} :	consumer i's virtual price for buying good j
ξ^m_i :	consumer i's virtual price for money
η^l_{ij} :	firm j's virtual price for buying labor from consumer i
η^x_{kj} :	firm j's virtual price for buying good j
η^y_j :	firm j's virtual price for selling the good it produces
η^m_j :	firm j's virtual price for money
R_i :	agent i's virtual income
	Other variables
ts :	the trade surplus
pbd :	public budget deficit
r°_j :	firm j's revenues received at the beginning of the period
θ_{ji} :	consumer i's share of the profits in firm j
ϑ_{hij} :	independently distributed stochastic variables
$J_{ij}(k)$:	an indicator for which side of labor market ij is rationed in regime k
$I_j(k)$:	an indicator for which side of goods market j is rationed in regime k
P^l_{1ij} :	prob. density for the ϑ-s in labor market ij when supply is rationed
P^l_{2ij} :	prob. density for the ϑ-s in labor market ij when demand is rationed
P^x_{1j} :	prob. density for the ϑ-s in goods market j when supply is rationed
P^x_{2j} :	prob. density for the ϑ-s in goods market j when demand is rationed
P^*_{ij} :	probability that there is an interior solution in labor market ij
P^{**}_j :	probability that there is an interior solution in goods market j

relationships have not been modeled. We let Y_j denote the sum of firm j's production for exports and for investment. Total production is thereby given by $y_j + Y_j = y_j + y_{inv,j} + y_{\exp,j}$, where Y_j is exogenous. We denote the stock of capital in firm j at the beginning of the period as K_j°. It can either be considered a vector of the different types of capital goods bought by the firm (taking depreciation into account) or as an aggregate denoting the total production capital in the firm. It is in any case outside the scope of this paper to discuss in any detail the composition of the capital stock. Dividends to the consumers are based on last periods profits π_{jk}°.

The after-tax price of good j is denoted p_j, while the wage received by consumer i when working for firm j is denoted w_{ij}. One should note that wages are both firm and worker specific. We let v_{1j} denote the price of investment goods from firm j and v_{2j} denote the price received for exports (in the local currency). Since we assume that these prices are governed by longer term contracts, they are not necessarily equal to the product price p_j. All prices are assumed to be exogenous and do not necessarily clear the markets.

It is assumed that the government levies two types of taxes, one on labor income and one on commodities (production and consumption inputs) and hands out lump-sum subsidies. There is no tax on investment goods or on exports. The tax rate on agent j's labor income is denoted t_{1j}, the rate of commodity taxation on good j is denoted t_{2j}, and the transfer to agent j at the beginning of the period is denoted t_{3j}°. The central government sets the budget of the government firms through the transfer t_{3j}. It is also assumed that dividend payments to the consumer are based on last periods profits. For notational convenience we let the vector p^* be the vector of after-tax prices $[p_1, \ldots, p_M]$, $w_{i\cdot}$ be the vector of wages $[w_{i1}, \ldots, w_{iM}]$ faced by consumer i, and $w_{\cdot j}$ be the vector of wages $[w_{M+1\,j}, \ldots, w_{M+N\,j}]$ faced by firm j. Since private firms do not demand goods or services from the government firms, we split the tax-adjusted price vector in two, $p^* = [p_1^*, p_2^*]$, where $p_1^* = [p_1, \ldots, p_{M_2}]$ and $p_2^* = [p_{M_2+1}, \ldots, p_M]$.

Money held over the production period by firm j is denoted as m_{fj} and money held by consumer i is denoted as m_{ci}. We let m_{fj}° and m_{ci}° denote these money holdings in the previous period. At the end of each period the firms receive revenue from sales and the private firms distribute profits. The total stock of money held by agent j at the beginning of the period is denoted C_j°. This will for the different agents be

$$
\begin{aligned}
C_j^\circ &= m_{fj}^\circ + (1 - t_{2j}^\circ) p_j^\circ y_j^\circ + v_{1j}^\circ\, y_{inv,j}^\circ + v_{2j}^\circ\, y_{exp,j}^\circ - \pi_j^\circ + t_{3j}^\circ, \quad j = {}_{1,\ldots,M_1}, \\
C_j^\circ &= m_{fj}^\circ + (1 - t_{2j}^\circ) p_j^\circ y_j^\circ + v_{1j}^\circ\, y_{inv,j}^\circ - \pi_j^\circ + t_{3j}^\circ, \qquad j = {}_{M_1+1,\ldots,M_2}, \\
C_j^\circ &= m_{fj}^\circ + (1 - t_{2j}^\circ) p_j^\circ y_j^\circ + t_{3j}^\circ, \qquad j = {}_{M_2+1,\ldots,M}, \\
C_j^\circ &= m_{ci}^\circ + \sum_j \theta_{ji} \pi_j^\circ + t_{3j}^\circ, \qquad j = {}_{M+1,\ldots,M+N},
\end{aligned}
$$

where $(1 - t^{\circ}_{2j})p^{\circ}_j$ is the price before taxes have been added. The price p°_j is thereby the after-tax price of a good and $1/((1 - t^{\circ}_{2j})$ the tax rate applied to the before-tax price received by the producer. The total stock of money each consumer i has at the end of the period is equal to the money held during the previous period plus the consumer's share of profits in the M_2 firms plus lump- sum transfers from the government. Each firm j holds money held during the previous period plus income received from sales minus profits paid to consumers (π°_j) plus subsidies from the government. As can be seen from the equations above, the main differences between private and government firms is that the latter do not pay out profits, produce investment goods, or produce for export. In addition we have that only consumers purchase goods from the government firms.

Since prices and wages do not clear markets, the firms and consumers may be rationed. Benassy (1975) introduced the concept of *a rationing mechanism* which expresses an agent's transactions as a function of the actions undertaken by the agent and the information he has. The rationing mechanism must be such that the net transactions of all the agents are consistent with each other. Examples of rationing mechanisms are uniform rationing where all the agents face the same rations and proportional rationing where the rations are proportional to the expressed demands and supplies. As mentioned, we assume in the following that the economy at any time is in a Drèze equilibrium. This implies that the information each agent has consists of the prices, wages, and the quantity constraints the agent faces in all markets. These quantity constraints consist of an upper bound $\overline{x}_{ji}$ or $\overline{l}_{ij}$ and a lower bound $\underline{x}_{ji}$ or $\underline{l}_{ij}$ for the net purchase of goods or labor respectively. In the same manner $-\underline{y}_j$ is the upper constraint on firm j's net sale of good j for use as a consumption or production input. Note that this constraint does not include production for investment purposes or for export as these are exogenously given. For notational convenience we let $\overline{x}_{\cdot j}$ denote the vector of upper constraints $\overline{x}_{k1}$ faced by agent j in the goods market, $\overline{l}_{\cdot j}$ denote the vector of upper constraints $\overline{l}_{ij}$ faced by firm j in the labor market, and $\underline{l}_{i\cdot}$ the vector of lower constraints $\underline{l}_{ij}$ faced by consumer i in the labor market.

The agents' actions consist of expressing their effective demands and supplies to the markets. *Effective demands and supplies* are such that they take into account the information the agents have about rationing and how the rationing will affect them through the rationing mechanism. In a Drèze equilibrium it is assumed that these effective demands and supplies are Drèze demands and supplies. *Drèze demands and supplies* are the result of maximizing utility or profits subject to the budget constraint and all quantity constraints that exist. A shortcoming of this type of effective demand is that it does not send a signal to the markets of the degree of rationing faced by the agents. The concept of a Drèze equilibrium does not specify how the quantity constraints are distributed among agents. In the following these are assumed to be latent. We let y^s_j denote firm j's Drèze supply of good j as a consumption or production input, x^d_{ji} agent i's Drèze demand

for good j as a consumption or production input, l^s_{ij} consumer i's Drèze supply of labor to firm j, l^d_{ij} firm j's Drèze demand for consumer i's labor, m^d_{fj} firm j's Drèze demand for money, and m^d_{ci} consumer i's Drèze demand for money. It is important to note that the Drèze demands and supplies will equal the observed transactions in a Drèze equilibrium. In the following we will therefore let variables denoting these also denote the transacted quantites. The only exception is the labor market where we will sometimes denote transacted labor by $l^*_{ij} = l^s_{ij} = l^d_{ij}$. We let $\mathbf{x}^d_i$ be a vector of agent i's Drèze demands for goods, $\mathbf{l}^d_i$ be a vector of the firm i's demand for labor, and $\mathbf{l}^s_i$ be a vector of consumer i's supply of labor, with

$$\begin{aligned}
\mathbf{x}^d_i &= \begin{cases} \left[x^d_{1i}, \cdots, x^d_{i-1\,i}, x^d_{i+1\,i}, \cdots, x^d_{M_2 i}\right] & \text{when } \ j = {}_{1,\ldots,M} \\ \left[x^d_{1i}, \cdots, x^d_{Mi}\right] & \text{when } \ j = {}_{M+1,\ldots,M+N}, \end{cases} \\
\mathbf{l}^d_i &= \left[l^d_{M+1\,i}, \cdots, l^d_{M+N\,i}\right], \qquad j = {}_{1,\ldots,M}, \\
\mathbf{l}^s_i &= \left[l^s_{i1}, \cdots, l^s_{iM}\right], \qquad j = {}_{M+1,\ldots,M+N}.
\end{aligned}$$

For a more detailed discussion of different types of equilibrium, rationing mechanisms, and effective demands see for example Benassy (1975), Böhm (1989), or Andreassen (1993).

2.1 Utility maximization

We assume that workers have preferences both over how many hours a year they work and where they work (their disutility differs according to firms). That individuals have preferences for where they work might reflect the different working conditions in the different firms or the location of the firm in relationship to the worker. In the same manner the firms look upon each worker as a separate input. Let $U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)$ be a utility function which is continuously differentiable, increasing in x_{ji} and m_{ci}, decreasing in l_{ij} and strictly quasi-concave. The quasi-concavity of the utility function implies that the consumer prefers to consume a variety of commodities rather that to consume any one commodity. A convex combination of any two labor bundles (with positive weights) is preferred to either labor bundle alone. In other words we make the rather unrealistic assumption that the consumers would prefer working many places to working in only one place.

Consumer i's holding of the numeraire good money is denoted by m_{ci} (end-of-period balance). Money enters the utility function as the only means for the consumer to transfer purchasing power between periods (besides stock ownership, which is exogenous). The utility function can therefore be interpreted as an indirect utility function taking into account intertemporal budget constraints. Another interpretation of the utility function is to view it as a derived utility function into which the consumer's transactions technology has been absorbed, see Feenstra (1986) and Samuelson and Sato (1984). Money holdings are assumed to always be positive.

The budget constraint for individual i is

$$-\sum_{j=1}^{M}(1-t_{1i})w_{ij}l_{ij}+\sum_{j=1}^{M}p_jx_{ji}+m_{ci} \quad = \quad m_{ci}^{\circ}+\sum_{j=1}^{M_2}\theta_{ji}\pi_j^{\circ}+t_{3i}^{\circ}. \tag{1}$$

The traded quantities are Drèze demands and supplies x_{ji}^d, l_{ij}^s, and m_{ci}^d resulting from consumer i maximizing the utility function

$$U_i(m_{ci},\mathbf{x}_i,\mathbf{l}_i) \tag{2}$$

with respect to $m_{ci}, x_{1i},\cdots,x_{Mi}, l_{i1},\cdots,l_{iM}$ subject to

$$\begin{aligned}
C_i^{\circ} &= -\sum_{j=1}^{M}(1-t_{1i})w_{ij}l_{ij}+\sum_{j=1}^{M}p_jx_{ji}+m_{ci},\\
\sum_{j=1}^{M}l_{ij} &\leq \bar{L},\\
0 \leq x_{ji} &\leq \overline{x}_{ji}, \quad j=1,\ldots,M,\\
0 &\leq x_{ji}, \quad j=1,\ldots,M_1,\\
\underline{l}_{ij} \leq -l_{ij} &\leq 0, \quad j=M_1+1,\ldots,M,\\
m_{ci} &> 0,
\end{aligned} \tag{3}$$

where $\bar{L}$ is the maximum number of hours it is possible to work. We assume that in practice the constraint $\sum_{j=1}^{M}l_{ij}\leq\bar{L}$ is never binding (nobody works 24 hours a day). This allows us to ignore this constraint in the following. Note that the consumer is never rationed in the market for tradeable goods, since any surplus demand can be met by imports.

Utility maximization yields the Drèze demands and supplies

$$l_{ij}^s = S_{l_{ij}}\left(\overline{x}_{\cdot i},\underline{l}_{i\cdot},p^*,w_{i\cdot},t_{i1},C_i^{\circ},\bar{L}\right), \quad j=1,\ldots,M \tag{4}$$

$$x_{ji}^d = D_{x_{ji}}\left(\overline{x}_{\cdot i},\underline{l}_{i\cdot},p^*,w_{i\cdot},t_{i1},C_i^{\circ},\bar{L}\right), \quad j=1,\ldots,M \tag{5}$$

and

$$m_{ci}^d = D_{m_{ci}}\left(\overline{x}_{\cdot i},\underline{l}_{i\cdot},p^*,w_{i\cdot},t_{i1},C_i^{\circ},\bar{L}\right) \tag{6}$$

for each consumer i. Under the assumption that $U(m_{ci},\mathbf{x}_i\mathbf{l}_i)$ is strictly quasi- concave the above problem has a unique solution.

2.2 Profit maximization

Private and government firms maximize profits. We denote firm j's production function $F_j(m_{fj},\mathbf{x}_j,\mathbf{l}_j,K_j^{\circ})$. The production function is continuously differentiable, increasing in the arguments, and

strictly concave. Money enters the production function because it is assumed that money holdings are needed for transaction purposes thereby facilitating production and as a means of transferring purchasing power between periods. The above production function absorbs the firm's transaction technologies in factor markets into the production function. The existence of such a derived production function can be analyzed in the same manner as the analysis of including money in the utility function in Feenstra (1986) and Samuelson and Sato (1984). See Barnett (1987) for a discussion of monetary aggregation theory under the assumption that money balances enter both the utility and the production function.

Generally one might hypothesize that the private firms objectives are more complex than just maximizing short term profits. For example Heller and Star (1979) consider the firms's short term objective function to be a function of short run profits, inventory holdings and capital accumulation, dividends, and retained earnings. Such a short term objective function can be thought of as representing the reduced form of the firm's intertemporal maximization problem taking into account limited information and incomplete markets. In such a context assuming that the private firms are constrained by a budget constraint such as the one above does not seem an implausible assumption.

The assumption that the private firms are constrained by sales in the previous period ensures that the production possibility set is bounded. The budget constraint for private firm j is

$$\begin{aligned} C_j^\circ &= m_{fj}^\circ + r_j^\circ - \pi_j^\circ + t_{3j}^\circ \\ &= \sum_{i=M+1}^{M+N} w_{ij} l_{ij} + \sum_{k=1\, k\neq j}^{M_2} p_k x_{kj} + m_{fj} + \sum_{k=1}^{M_2} v_{1k}\, inv_{kj} \end{aligned} \tag{7}$$

and for government firm j

$$\begin{aligned} C_j^\circ &= m_{fj}^\circ + r_j^\circ + t_{3j}^\circ \\ &= \sum_{i=M+1}^{M+N} w_{ij} l_{ij} + \sum_{k=1\, k\neq j}^{M_2} p_k x_{kj} + m_{fj} + \sum_{k=1}^{M_2} v_{1k}\, inv_{kj}, \end{aligned} \tag{8}$$

where $r_j^\circ = (1-t_{2j}^\circ)p_j^\circ y_j^\circ + v_{1j}^\circ\, y_{inv,j}^\circ + v_{2j}^\circ\, y_{exp,j}^\circ$ denotes the firm's revenues in the previous period. Note that government firms do not pay out profits to the consumers.

The traded quantities are the Drèze demands y_j^s, x_{kj}^d, l_{ij}^d, and m_{fj}^d resulting from firm j's maximizing profits, π_j,

$$\begin{aligned} \pi_j &= (1-t_{2j})p_j y_j + v_{1j}\, y_{inv,j} + v_{2j}\, y_{exp,j} - \sum_{i=M+1}^{M+N} w_{ij} l_{ij} - \sum_{k=1\, k\neq j}^{M_2} p_k x_{kj} \\ &\quad -(m_{fj} - m_{fj}^\circ) \end{aligned} \tag{9}$$

with respect to $m_{fj}, x_{1j}, \ldots, x_{j-1\,j}, x_{j+1\,j}, \ldots, x_{M_2 j}, l_{M+1\,j}, \cdots, l_{M+N\,j}$ subject to:

$$
\begin{aligned}
C_j^\circ - \sum_{k=1}^{M_2} v_{1k}\, inv_{kj} &= \sum_{i=M+1}^{M+N} w_{ij} l_{ij} + \sum_{k=1\, k\neq j}^{M_2} p_k x_{kj} + m_{fj}, \\
y_j &= -y_j, \\
\underline{y}_j \;\leq\; -F_j(m_{fj}, \mathbf{x}_j, \mathbf{l}_j, K_j^\circ) - Y_j &\leq 0, \\
0 &\leq x_{kj}, \quad k = {}_{1,\ldots,M_1}, \;\; k \neq j, \\
0 \;\leq\; x_{kj} &\leq \overline{x}_{kj}, \quad k = {}_{M_1+1,\ldots,M_2}, \;\; k \neq j, \\
0 \;\leq\; l_{ij} &\leq \bar{l}_{ij}, \quad i = {}_{M+1,\ldots,M+N}, \\
m_{fj} &> 0.
\end{aligned} \tag{10}
$$

Profit maximization yields the Drèze demands and supplies

$$
l_{ij}^d = D_{l_{ij}}\left(\underline{y}_j, \overline{x}_{\cdot j}, \bar{l}_{\cdot j}, p_1^*, w_{\cdot j}, t_{2j}, C_j^\circ, Y_j\right), \quad i = {}_{M+1,\ldots,M+N}, \tag{11}
$$

$$
x_{kj}^d = D_{x_{kj}}\left(\underline{y}_j, \overline{x}_{\cdot j}, \bar{l}_{\cdot j}, p_1^*, w_{\cdot j}, t_{2j}, C_j^\circ, Y_j\right), \quad k = {}_{1,\ldots,M}, \;\; k \neq j, \tag{12}
$$

$$
m_{fj}^d = D_{m_{fj}}\left(\underline{y}_j, \overline{x}_{\cdot j}, \bar{l}_{\cdot j}, p_1^*, w_{\cdot j}, t_{2j}, C_j^\circ, Y_j\right), \tag{13}
$$

and

$$
y_j^s = F\left(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d, K_j^\circ\right) - Y_j \tag{14}
$$

for each firm j. Under the assumptions that $F(m_{fj}, \mathbf{x}_j,\ \mathbf{l}_j,\ K_j^\circ)$ is strictly concave the above problem has a unique solution. For government firms we have that $Y_j = y_{inv,j} + y_{exp,j} = 0$.

2.3 Taxes, the public budget deficit and the trade surplus

Imports, the trade surplus, tax revenue, the public budget deficit, and changes in the money supply are endogenous in the model. Total tax revenue for the government, T, is

$$
T = \sum_{j=1}^{M} \sum_{i=M+1}^{M+N} t_{1i} w_{ij} l_{ij} + \sum_{j=1}^{M_2} \sum_{i=1\, i\neq j}^{M} t_{2j} p_j x_{ji} + \sum_{j=M_2+1}^{M} \sum_{i=M+1}^{M+N} t_{2j} p_j x_{ji} \tag{15}
$$

and total subsidies are $\sum_{i=1}^{M+N} t_{3i}$. The government decides exogenously the price and volume of it's supply of goods and services to the consumers. The resulting public budget deficit, *pbd*, can be written as

$$
pbd = -T + \sum_{i=1}^{M+N} t_{3i}, \tag{16}
$$

where $\sum_{i=1}^{M_2} t_{3i}$ is the total subsidies to private firms, $\sum_{i=M_2+1}^{M} t_{3i}$ is the total transfer to the government firms, and $\sum_{i=M+1}^{M+N} t_{3i}$ is the total transfers to the consumers.

The domestic demand for tradeable good $j \in \{1, \ldots, M_1\}$ for use as a consumption or production input and for investment leads to imports

$$imp_j = \sum_{k=1}^{M} inv_{jk} + \sum_{i}^{M+N} x_{ji} - y_j - y_{inv,j}.$$

Short term production y_j can not be greater than the short term use of goods $\sum x_{ij}$,

$$imp_j - \sum_{k=1}^{M} inv_{jk} + y_{inv,j} = \sum_{i}^{M+N} x_{ji} - y_j \ \geq \ 0.$$

Any short term surplus demand is met by imports while short term surplus supply will imply rationing. This non-symmetry in the assumptions about imports and exports is done to take into account that it is often difficult for firms to quickly switch production from domestic to foreign markets. Since there is no import of products $M_1 + 1, \ldots, M$, we have that:

$$y_{inv,j} = \sum_{k} inv_{jk}, \quad j = {}_{M_1+1,\ldots,M_2},$$

and

$$y_j = \sum_{k} x_{jk}, \quad j = {}_{M_1+1,\ldots,M}.$$

Equation (19) only covers the goods $M_1 + 1, \ldots, M_2$ since government firms do not produce goods which can be used for capital investment.

The trade surplus ts is the difference between the value of the production in the economy minus the value of the goods used in the economy,

$$ts \quad = \quad v_{1j}(y_{inv,j} - \sum_{i=1}^{M} inv_{ji}) + \sum_{j=1}^{M_1} \left(v_{2j}\, y_{exp,j} + p_j(y_j - \sum_{i=1}^{M+N} x_{ji}) \right). \tag{17}$$

Since there are no financial markets in our model, the public budget deficit and the trade surplus must be financed by money. We have earlier assumed that profits are first distributed in the period after that in which they have been earned. This implies that in each period there will be a stock of undistributed profits which will be a component of the total money stock during that

period. The total stock of money at the beginning of the period, $\bar{m}^\circ$ will be:

$$\begin{aligned}\bar{m}^\circ &= \sum_{j=1}^{M+N} C_j^\circ \\ &= \sum_{j=1}^{M_2}(m_{fj}^\circ + r_j^\circ - \pi_j^\circ + t_{3j}^\circ) + \sum_{j=M_2+1}^{M}(m_{fj}^\circ + r_j^\circ + t_{3j}^\circ) \\ &\quad + \sum_{j=M+1}^{M+N}(m_{cj}^\circ - \sum_{k=1}^{M_2}\theta_{kj}\pi_k^\circ + t_{3j}).\end{aligned} \tag{18}$$

From the above equation we have that the net acquisition of money balances $\Delta\bar{m}$ from on period to the next will be

$$\begin{aligned}\Delta\bar{m} &= \bar{m} - \bar{m}^\circ \\ &= \sum_{j=1}^{M}(r_j - r_j^\circ) + \sum_{j=1}^{M}(m_{fj} - m_{fj}^\circ) + \sum_{i=M+1}^{M+N}(m_{ci} - m_{ci}^\circ) + \sum_{j=1}^{M+N}(t_{3j} - t_{3j}^\circ) \\ &= pbd + ts,\end{aligned} \tag{19}$$

where we derive the last expression by substituting for $m_{ci} - m_{ci}^\circ - t_{3j}^\circ$ from the budget constraint for the consumer and for $m_{fi} - r_j^\circ - m_{fi}^\circ - t_{3j}^\circ$ from the budget constraint for the firms.

The aggregate money stock held during the period must equal the aggregate money stock at the beginning of the period minus taxes and an eventual trade deficit (the transfers t_{3i} are paid at the end of the period). We therefore have that

$$\begin{aligned}\sum_{j=1}^{M} m_{fj}^d + \sum_{i=M+1}^{M+N} m_{ci}^d &= \sum_{j=1}^{M+N} C_j^\circ - T + ts \\ &= \bar{m}^\circ - T + ts.\end{aligned} \tag{20}$$

Neary (1980) discusses a similar open economy model with a representative household, two production sectors, and a government sector. One production sector produces a traded good and the other a non-traded good. One of the main results in Neary's paper is that in a situation where the wage and the price of the non-traded good are sticky a wage cut may not increase employment and a devaluation may not improve the trade balance.

2.4 Rationing (Drèze) equilibria

We are now able to prove that the above model is such that there exists a Drèze equilibrium. Letting $\{p_j\}$ denote the vector of all prices and using similar notation for the other variables we have that the following theorem applies.

Theorem 1 *For any non-negative* $(\{p_j\},\{w_{ij}\},\{t_{1i}\},\{t_{2j}\},\ \{t_{3i}\},\ \{K_i^\circ\},\ \{C_i^\circ\},\ \bar{L},\ \{Y_j\},\ \{inv_{ij}\})$ *such that* $p_j > 0,\ 1 > t_{2j} \geq 0$ *for all* $j,\ 1 > t_{1i} \geq 0$, *for all* i *and* $w_{ij} > 0$ *for all combinations of* $j = {}_{1,...,M}$ *and* $i = {}_{M+1,...,M+N}$, *there exist maximum and minimum constraints* $(\{\underline{y}_i\},\{\bar{l}_{ij}\},\{\underline{l}_{ij}\},\{\overline{x}_{kj}\})$ *satisfying*

1.1
$$\underline{y}_j \leq 0, \qquad j = {}_{1,...,M}$$
$$0 \leq \bar{l}_{ij}, \qquad i = {}_{M+1,...,M+N},\ \text{for all } j = {}_{1,...,M}$$
$$\underline{l}_{ij} \leq 0, \qquad j = {}_{1,...,M}\ \text{for all } i = {}_{M+1,...,M+N},$$
$$0 \leq \overline{x}_{kj}, \qquad k = {}_{M_1+1...,M}\ \text{for all } j = {}_{1,...,M+N},\ \ k \neq j$$

1.2
$$y_j^s - \textstyle\sum_{i=1}^{M+N} x_{kj}^d = 0, \qquad j = {}_{M_1+1,...,M}$$
$$l_{ij}^s - l_{ij}^d = 0, \qquad i = {}_{M+1,...,M+N}, \quad j = {}_{1,...,M}$$
where $x_{1i}^d,\ldots,x_{Mi}^d,\ l_{i1}^s,\ldots,l_{iM}^s$ *for* $i = {}_{M+1,...,M+N}$ *are the Drèze demands and supplies which solve the problem*

$$\begin{aligned}
\max \quad & U_i(m_{ci},\mathbf{x}_i,\mathbf{l}_i) \\
s.t. \quad & -\sum_{j=1}^{M}(1-t_{1i})w_{ij}l_{ij} + \sum_{j=1}^{M} p_j x_{ji} + m_{ci} \ = \ C_j^\circ, \\
& \sum_{j=1}^{M} l_{ij} \ \leq \ \bar{L}, \\
& 0 \ \leq \ x_{ji}, \quad j = {}_{1,...,M_1}, \\
& 0 \ \leq \ x_{ji} \ \leq \ \overline{x}_{ji}, \quad j = {}_{M_1+1,...,M}, \\
& \underline{l}_{ij} \ \leq \ -l_{ij} \ \leq \ 0, \quad j = {}_{1,...,M}, \\
& m_{ci} \ > \ 0,
\end{aligned}$$

and where $x_{1j}^d,\ldots,x_{Mj}^d,\ l_{M+1\,j}^d,\ldots,l_{M+N\,j}^d$ *for* $j = {}_{1,...,M}$ *are the Drèze demands and supplies which solve the problem*

$$\begin{aligned}
\max \quad & \pi_j \ = \ (1-t_{2j})p_j y_j - \sum_{i=1}^{N} w_{ij}l_{ij} - \sum_{k=0\,k\neq j}^{M-1} p_k y_{kj} - (m_{fj} - m_{fj}^\circ) \\
s.t. \quad & \sum_{i=M+1}^{M+N} w_{ij}l_{ij} + \sum_{k=1\,k\neq j}^{M_2} p_k x_{kj} + m_{fj} \ = \ C_j^\circ - \sum_{k=1}^{M_2} v_{1k}\, inv_{kj} \\
& 0 \ \leq \ F_j(m_{fi},\mathbf{x}_j,\mathbf{l}_j,K_j^\circ) - Y_j \ \leq \ -\underline{y}_k \\
& 0 \ \leq \ x_{kj}, \qquad k = {}_{1,...,M_1},\ \ k \neq j \\
& 0 \ \leq \ x_{kj} \ \leq \ \overline{x}_{jk}, \quad k = {}_{M_1+1,...,M},\ \ k \neq j, \\
& 0 \ \leq \ l_{ij} \ \leq \ \bar{l}_{ik}, \quad i = {}_{M+1,...,M+N}, \\
& m_{fj} \ > \ 0.
\end{aligned}$$

1.3 *1.* $-y_j^s = \underline{y}_j$ *for some* j *implies that* $x_{ji}^d < \overline{x}_{ji}$ *for all* i*;*

2. $x_{ji}^d = \overline{x}_{ji}$ *for some* i *implies that* $-y_j^s > \underline{y}_j$*;*

3. $-l_{ij}^s = \underline{l}_{ij}$ *implies that* $l_{ij}^d < \bar{l}_{ij}$*;*

4. $l_{ij}^d = \bar{l}_{ij}$ *implies that* $-l_{ij}^s > \underline{l}_{ij}$*.*

The constraints $(\{\underline{y}_i\}, \{\bar{l}_{ij}\}, \{\underline{l}_{ij}\}, \{\overline{x}_{kj}\})$ constitute a Drèze equilibrium at $(\{p_j\}, \{w_{ij}\}, \{t_{1i}\}, \{t_{2j}\}, \{t_{3i}\}, \{K_i^\circ\}, \{C_i^\circ\}, \bar{L}, \{Y_j\}, \{inv_{ij}\})$. In most cases such an equilibrium will not be unique and there will exist many Drèze equilibria for a given set of exogenous variables. Note that there are no constraints on the use of goods $1, \ldots, M_1$ which can be imported. Proof of theorem 1 is given in appendix A. It borrows heavily from the proof in Mukherji, Anjan (1990) pp. 153-157, which is a modified version of the well known result first formulated by Drèze (1975).

When we later introduce an econometric specification it is important to keep in mind that the ensuing likelihood function must be well defined. This is the same as requiring the model to be *coherent*, making it possible to infer the distribution of the observed variables from the stochastic specification. Uniqueness of the Drèze equilibrium would guarantee that the likelihood function is well defined, but as noted above, this will in general not be the case.

3 Virtual prices

In the following we derive the inverse demand and supply functions associated with the model and discuss how the assumption of a Drèze equilibrium implies certain relationships between these in each market. These relationships are simpler for the labor market than the goods markets due to our assumption that each combination of firm and consumer constitutes a separate labor market.

3.1 Using virtual prices to describe the agents' behavior

The Lagrange equation for the consumer i's maximization problem is

$$\begin{aligned}\mathcal{L}(m_{ci}, \mathbf{x}_i, \mathbf{l}_i) &= U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i) - \lambda_{ci}\left(\sum_{j=1}^{M} p_j x_{ji} - (1-t_{1i})\sum_{j=1}^{M} w_{ij} l_{ij} + m_{ci} - C_i^\circ\right) \\ &\quad -\mu\left(\sum_{j=1}^{M} l_{ij} - \bar{L}\right) - \sum_{j=1}^{M} \phi_{l_{ij}}\left(l_{ij} + \underline{l}_{ij}\right) - \sum_{j=M_1+1}^{M} \phi_{x_{ij}}\left(x_{ij} - \overline{x}_{ij}\right), \end{aligned} \tag{21}$$

where $\mathcal{L}(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)$ is the Lagrange functions. Note that $\underline{l}_{ij}$ is a negative variable. The Lagrange multiplier λ_{ci} is assumed to be positive and the Lagrange multipliers μ, and the ϕ-s are assumed to be non-negative.

The solution to the constrained optimization problem can be characterized by the Kuhn-Tucker conditions:

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial l_{ij}} = \frac{\partial U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)}{\partial l_{ij}} + \lambda_{ci}(1 - t_{1i})w_{ij} - \phi_{l_{ij}} - \mu \leq 0, \\ &\frac{\partial \mathcal{L}}{\partial l_{ij}} l_{ij} = 0, \end{aligned}\right\} \quad j = 1,\ldots,M \tag{22}$$

$$\frac{\partial \mathcal{L}}{\partial x_{ji}} = \frac{\partial U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)}{\partial x_{ji}} - \lambda_{ci} p_j = 0, \qquad j = 1,\ldots,M_1 \tag{23}$$

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial x_{ji}} = \frac{\partial U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)}{\partial x_{ji}} - \lambda_{ci} p_j - \phi_{x_{ji}} \leq 0, \\ &\frac{\partial \mathcal{L}}{\partial x_{ji}} x_{ji} = 0, \end{aligned}\right\} \quad j = M_1+1,\ldots,M \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial m_{ci}} = \frac{\partial U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)}{\partial m_{ci}} - \lambda_{ci} = 0, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{ci}} = \sum_{j=1}^{M} p_j x_{ji} - (1 - t_{1i}) \sum_{j=1}^{M} w_{ij} l_{ij} + m_{ci} - C_i^{\circ} = 0, \tag{26}$$

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \mu} = \sum_{j=1}^{M} l_{ij} - \bar{L} \leq 0, \\ &\frac{\partial \mathcal{L}}{\partial \mu} \mu = 0, \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \phi_{l_{ij}}} = l_{ij} + \underline{l}_{ij} \leq 0, \\ &\frac{\partial \mathcal{L}}{\partial \phi_{l_{ij}}} \phi_{l_{ij}} = 0, \end{aligned}\right\} \quad j = 1,\ldots,M \tag{28}$$

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}}{\partial \phi_{x_{ji}}} = x_{ji} - \overline{x}_{ji} \leq 0, \\ &\frac{\partial \mathcal{L}}{\partial \phi_{x_{ji}}} \phi_{x_{ji}} = 0. \end{aligned}\right\} \quad j = M_1+1,\ldots,M \tag{29}$$

If we assume local non-satiation ($\partial U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i)/\partial x_{ji} > 0$ for some j) then equations (26) to (33) are necessary and sufficient for a unique global optimum (Takayama (1985) p. 114 and pp. 135-137). As mentioned before we assume that the constraint $\bar{L}$ is so high (so many hours a year) that it is not binding for any individual and therefore $\mu = 0$.

We now describe the situation with rationing using virtual prices. We define ξ^l_{ij} as the virtual wage for labor supplied to firm j, ξ^x_{ji} as the virtual price for the good supplied by firm j, and ξ^m_i as the virtual price for money (which as numeraire is always equal to 1). Virtual prices in terms of the money numeraire can be defined as

$$\begin{aligned}\xi^l_{ij}(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i) &= -\frac{\partial U_i(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i)/\partial l_{ij}}{\partial U_i(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i)/\partial m_{ci}}, \\ &= (1 - t_{1i})w_{ij} - \phi_{l_{ij}}/\lambda_{ci} \qquad (30)\end{aligned}$$

$$\begin{aligned}\xi^x_{ji}(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i) &= \frac{\partial U_i(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i)/\partial x_{ji}}{\partial U_i(m^d_{ci}, \mathbf{x}^d_i, \mathbf{l}^s_i)/\partial m_{ci}}, \\ &= p_j + \phi_{x_{ji}}/\lambda_{ci} \qquad (31)\end{aligned}$$

and

$$\xi^m_i = 1, \qquad (32)$$

which are the prices which support the Drèze demands and supplies, m^d_{ci}, $\mathbf{x}^d_i$, and $\mathbf{l}^s_i$ as an unconstrained utility maximization solution given a virtual income R_i. This unconstrained utility maximization satisfies the budget constraint

$$m^d_{ci} + \sum_{j=1}^{M} \xi^x_{ji} x^d_{ji} - \sum_{j=1}^{M} \xi^l_{ij} l^s_{ij} = R_i \qquad (33)$$

and by substituting this constraint into the original constraint we get that the relationship between virtual income R_i and nominal income C°_i is

$$R_i = C^\circ_i + \sum_{j=1}^{M} (\xi^x_{ji} - p_i) x^d_{ji} + \sum_{j=1}^{M} ((1 - t_{1i})w_{ij} - \xi^l_{ij}) l^s_{ij}. \qquad (34)$$

In the case of non-rationed goods the virtual prices will be equal to the observed prices. For a discussion of virtual prices see Deaton and Muellbauer (1980) pp. 109-114. A more detailed discussion of the use of virtual prices in econometric disequilibrium models can be found in Lee (1986). Note that the virtual price of labor is defined as the gain in utility from working one less marginal unit of time while the virtual price of a consumer good is defined as the gain in utility from consuming an extra marginal unit.

The Lagrangan of the firm j's maximization problem is

$$\begin{aligned}
&\mathcal{L}(\dagger_{|}, \Updownarrow_{\{|}, \mathbf{x}_{|}, \mathbf{l}_{|}) \;=\; \\
&\quad (1-t_{2j})\pi_j - \lambda_{f1j}\left(\sum_{i=M+1}^{M+N} w_{ij}l_{ij} + \sum_{k=1\,k\neq j}^{M_2} p_k x_{kj} + m_{fj} - C_j^{\circ} + \sum_{k=1}^{M_2} v_{1k}\, inv_{kj}\right) \\
&\quad -\lambda_{f2j}\left(F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j) - 0\right) - \delta_y\left(y_j + \underline{y}_j\right) - \sum_{j=M+1}^{M+N} \delta_{l_{ij}}\left(l_{ij} - \bar{l}_{ij}\right) \\
&\quad -\delta_{x_{kj}} \sum_{k=M_1+1,\; k\neq j}^{M} \left(x_{kj} - \overline{x}_{kj}\right), \qquad (35)
\end{aligned}$$

where

$$F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j) = y_j + Y_j - F_j(m_{fj}, \mathbf{x}_j, \mathbf{l}_j, K_j^{\circ}).$$

The Lagrange multipliers λ_{f1j} and λ_{f2j} are assumed to be positive while the δ's are assumed to be non-negative.

The solution to the firm j's constrained optimization problem can be characterized by the Kuhn-Tucker conditions in the same manner as in the case of the consumers.

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial l_{ij}} = -w_{ij} - \lambda_{f1j} w_{ij} - \lambda_{f2j} \dfrac{\partial F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j)}{\partial l_{ij}} - \delta_{l_{ij}} \leq 0, \\ \dfrac{\partial \mathcal{L}}{\partial l_{ij}} l_{ij} = 0,\end{array}\right\} \; i = {}_{M+1,\ldots,M+N} \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial x_{kj}} = -p_k - \lambda_{f1j} p_k - \lambda_{f2j} \frac{\partial F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j)}{\partial x_{kj}} = 0, \qquad k = {}_{1,\ldots,M_1} \tag{37}$$

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial x_{kj}} = -p_k - \lambda_{f1j} p_k - \lambda_{f2j} \dfrac{\partial F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j)}{\partial x_{kj}} - \delta_{x_{kj}} \leq 0, \\ \dfrac{\partial \mathcal{L}}{\partial x_{kj}} x_{kj} = 0,\end{array}\right\} \; k = {}_{M_1+1,\ldots,M_2} \tag{38}$$

$$\frac{\partial \mathcal{L}}{\partial m_{fj}} = -1 - \lambda_{fj} - \lambda_{f2j} \frac{\partial F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j)}{\partial m_{fj}} = 0, \tag{39}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{f1j}} = \textstyle\sum_{i=M+1}^{M+N} w_{ij} l_{ij} + \sum_{k=1\, k\neq j}^{M_2} p_k x_{kj} + m_{fj} z_j + \sum_{k=1}^{M_2} v_{1k}\, inv_{kj} - C_j^{\circ} = 0, \tag{40}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_{f2j}} = y_j + Y_j - F_j(m_{fj}, \mathbf{x}_j, \mathbf{l}_j, K_j^{\circ}) = F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j) = 0, \tag{41}$$

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial \delta_{l_{ij}}} = l_{ij} - \bar{l}_{ij} \leq 0, \\ \dfrac{\partial \mathcal{L}}{\partial \delta_{l_{ij}}} \delta_{l_{ij}} = 0,\end{array}\right\} \; i = {}_{M+1,\ldots,M+N} \tag{42}$$

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial \delta_{x_{kj}}} = x_{kj} - \overline{x}_{kj} \leq 0, \\ \dfrac{\partial \mathcal{L}}{\partial \delta_{x_{kj}}} \delta_{x_{kj}} = 0\end{array}\right\} \; k = {}_{M_1+1,\ldots,M} \tag{43}$$

$$\left.\begin{array}{l}\dfrac{\partial \mathcal{L}}{\partial \delta_y} = y_j + \underline{y}_j = 0, \\ \dfrac{\partial \mathcal{L}}{\partial \delta_y} \delta_y = 0,\end{array}\right\} \tag{44}$$

$$\frac{\partial \mathcal{L}}{\partial y_j} = (1 - t_{2j}) p_j - \lambda_{f2j} \frac{\partial F_j^*(y_j, m_{fj}, \mathbf{x}_j, \mathbf{l}_j)}{\partial y_j} - \delta_y = 0, \tag{45}$$

One should note that we assume technical efficiency by assuming that $y_j + Y_j = F_j(m_{fj}, \mathbf{x}_j, \mathbf{l}_j, K_j^{\circ})$.

The above conditions determine the firm's demand for inputs. The firm's production then follows from the production function. The Drèze supplies and demands y_j^s, x_{kj}^d, l_{ij}^d, and m_j^d are the optimal solution to the constrained profit maximization problem of firm j given the virtual income R_i defined in the same manner as for the consumer. We define η_{ij}^l as the virtual wage for labor supplied to firm j from consumer i, η_{kj}^x as the virtual price for the good supplied by firm k to firm j, η_j^m as the virtual price for money (which as numeraire is always equal to 1), and η_j^y as the virtual price for good produced by firm j. In the same manner as for the consumer, virtual prices (in terms of the money numeraire) can be defined as

$$\begin{aligned}
\eta_j^y(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d) &= -\frac{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial y_j^s}{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial m_{fj}}, \\
&= [(1-t_{2j})p_j - \delta_y]/(1+\lambda_{f1j}), \qquad (46)
\end{aligned}$$

$$\begin{aligned}
\eta_{ij}^l(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d) &= \frac{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial l_{ij}}{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial m_{fj}}, \\
&= w_{ij} + \delta_{l_{ij}}/(1+\lambda_{f1j}), \qquad (47)
\end{aligned}$$

$$\begin{aligned}
\eta_{kj}^x(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d) &= \frac{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial x_{kj}}{\partial F_j^*(y_j^s, m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d)/\partial m_{fj}}, \\
&= p_k + \delta_{x_{kj}}/(1+\lambda_{f1j}), \qquad (48)
\end{aligned}$$

and

$$\eta_j^m = 1, \qquad (49)$$

As long as a private firm's budget constraint or the rationing constraint is binding we have that $\eta_j^y(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d) > (1-t_{2j})p_j$. That this is the case when the budget constraint is binding ($\lambda_{f1j} > 0$) reflects the requirement that purchases of inputs be based on last years sales imposes an inefficiency on the firm. In the following we will not refer to a firm as rationed when it is only constrained by it's budget constraint, $\eta_j^y(m_{fj}^d, \mathbf{x}_j^d, \mathbf{l}_j^d) = (1-t_{2j})p_j/(1+\lambda_{f1j})$.

As an example of how the Kuhn-Tucker conditions correspond to the virtual prices consider

the following equations characterizing the consumer's supply of labor:

$$\xi^l_{ij}(\cdot) = (1-t_{1i})w_{ij} \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial l_{ij}} = \frac{\partial U_i(\cdot)}{\partial l_{ij}} + \lambda_{ci}(1-t_{1i})w_{ij} = 0, \tag{50}$$

$$\xi^l_{ij}(\cdot) < (1-t_{1i})w_{ij} \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial l_{ij}} = \frac{\partial U_i(\cdot)}{\partial l_{ij}} + \lambda_{ci}(1-t_{1i})w_{ij} - \phi_{l_{ij}} = 0, \tag{51}$$
$$\text{and} \quad \phi_{l_{ij}} > 0,$$

$$\xi^l_{ij}(\cdot) > (1-t_{1i})w_{ij} \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial l_{ij}} < 0 \ \text{ and } \ l_{ij} = 0. \tag{52}$$

From (55) it is apparent that if there is an interior solution with no rationing in the market for labor supplied by consumer i to firm j then the virtual price is equal to the wage (after taxes). From (56) we see that if the consumer is rationed in this micro labor market then the virtual price will be less than the wage. The last equation, (57), describes the situation when there is a corner solution. From this it follows that there are several ways in which quantities can be zero. The quantity used of a good may be rationed at zero level or it be zero as a result of a corner solution. There is also the special case when the agent's indifference curve is exactly tangent to the budget line at the point zero. From the equations defining the virtual price we see that ξ^l_{ij} is increasing in l^s_{ij}. The point where ξ^l_{ij} is such that the agent would like to supply exactly $l^s_{ij} = 0$ of labor is individual i's reservation wage. If the consumer chooses to work in firm j then the reservation wage is lower than the marginal wage at that firm. The virtual prices will of course depend on all other variables, for example all other wages. In the same manner we have that consumer i will not buy good j if the virtual price ξ^x_{ji} is lower than the marginal price p_j. Reasoning of this type can also be applied to the virtual prices of the firm.

We have earlier stated that the virtual prices can be viewed as the prices that would induce unrationed agents to demand and supply exactly the Drèze demands and supplies given a virtual income R_i. If we assume that consumer i is not rationed in any markets, the Lagrange multipliers $\phi_{l_{ij}}$ and $\phi_{x_{ij}}$ are zero. In this case solving the Kuhn-Tucker equations leads to the notional demand and supply functions for consumer i,

$$\tilde{l}^s_{ij} = \tilde{S}_{l_{ij}}(p_1, \ldots, p_M, {}_{(1-t_{1i})}w_{i1}, \ldots, {}_{(1-t_{1i})}w_{iM}, C^\circ_i), \quad j = {}_{1,\ldots,M}, \tag{53}$$

$$\tilde{x}^d_{ji} = \tilde{D}_{x_{ji}}(p_1, \ldots, p_M, {}_{(1-t_{1i})}w_{i1}, \ldots, {}_{(1-t_{1i})}w_{iM}, C^\circ_i), \quad j = {}_{1,\ldots,M}, \tag{54}$$

and

$$\tilde{m}^d_{ci} = \tilde{D}_{m_{ci}}(p_1, \ldots, p_M, {}_{(1-t_{1i})}w_{i1}, \ldots, {}_{(1-t_{1i})}w_{iM}, C^\circ_i).$$

In the same manner, if we assume that firm j is not rationed in any markets, we have that the

Lagrange multipliers $\delta_{l_{ij}}$, $\delta_{x_{kj}}$, and δ_y are zero. In this case solving the Kuhn-Tucker equations leads to the notional demand and supply functions for firm j,

$$\tilde{l}^d_{ij} = \tilde{D}_{l_{ij}}(p_1, \ldots, p_{M_2}, (1-t_{2j})p_j, w_{M+1\,j}, \ldots, w_{M+N\,j}, C^\circ_i), \quad j = {}_{M+1,\ldots,M+N}, \tag{55}$$

$$\tilde{x}^d_{kj} = \tilde{D}_{x_{kj}}(p_1, \ldots, p_{M_2}, (1-t_{2j})p_j, w_{M+1\,j}, \ldots, w_{M+N\,j}, C^\circ_i), \quad k = {}_{1,\ldots,M_2}, \tag{56}$$

and

$$\tilde{m}^d_{fj} = \tilde{D}_{m_{fj}}(p_1, \ldots, p_{M_2}, (1-t_{2j})p_j, w_{M+1\,j}, \ldots, w_{M+N\,j}, C^\circ_i).$$

One should note that the above functions admit the posibility of corner solutions. The Drèze demands and supplies of consumer i and firm j can now be written using the above notional demand functions. For consumer i we have

$$l^s_{ij} = \tilde{S}_{l_{ij}}(\xi^x_{1i}, \cdots, \xi^x_{Mi}, \xi^l_{i1}, \cdots, \xi^l_{iM}, R_i), \quad j = {}_{1,\ldots,M}, \tag{57}$$

$$x^d_{ji} = \tilde{D}_{x_{ji}}(\xi^x_{1i}, \cdots, \xi^x_{Mi}, \xi^l_{i1}, \cdots, \xi^l_{iM}, R_i), \quad j = {}_{1,\ldots,M}, \tag{58}$$

$$m^d_{ci} = \tilde{D}_{m_{ci}}(\xi^x_{1i}, \cdots, \xi^x_{Mi}, \xi^l_{i1}, \cdots, \xi^l_{iM}, R_i), \tag{59}$$

and for firm j

$$l^d_{ij} = \tilde{D}_{l_{ij}}(\eta^x_{1j}, \cdots, \eta^x_{M_2 j}, \eta^y_j, \eta^l_{M+1\,j}, \cdots, \eta^l_{M+N\,j}, R_i), \quad j = {}_{M+1,\ldots,M+N}, \tag{60}$$

$$x^d_{kj} = \tilde{D}_{x_{kj}}(\eta^x_{1j}, \cdots, \eta^x_{M_2 j}, \eta^y_j, \eta^l_{M+1\,j}, \cdots, \eta^l_{M+N\,j}, R_i), \quad j = {}_{1,\ldots,M_2}, \tag{61}$$

and

$$m^d_{fj} = \tilde{D}_{m_{fj}}(\eta^x_{1j}, \cdots, \eta^x_{M_2 j}, \eta^y_j, \eta^l_{M+1\,j}, \cdots, \eta^l_{M+N\,j}, R_i),$$

where

$$\begin{aligned}
\xi^x_{ji} &= p_j \text{ if the consumer isn't rationed in goods market } j,\\
\eta^x_{kj} &= p_k \text{ if the firm isn't rationed in goods market } k,\\
(1+\lambda_{f1j})\eta^y_j &= (1-t_{2j})p_k \text{ if the firm isn't rationed in goods market } k,\\
\xi^l_{ij} &= (1-t_{1i})w_{ij} \text{ if the consumer isn't rationed in labor market } ij,
\end{aligned}$$

and

$$\eta^l_{ij} = w_{ij} \text{ if the firm isn't rationed in labor market } ij.$$

An intuitively appealing measure of the spillover between markets is the difference between the agents' notional (Drèze) demand and supply functions and the notional demand functions. For

example the spillover from other markets to consumer i's demand for good j is according to this measure

$$\begin{aligned} x_{ji}^d - \tilde{x}_{ji}^d &= \tilde{D}_{x_{ji}}(\xi_{1i}^x, \cdots, \xi_{Mi}^x, \xi_{i1}^l, \cdots, \xi_{iM}^l, R_i) \\ &\quad -\tilde{D}_{x_{ji}}(p_1, \cdots, p_M, {}_{(1-t_{1i})}w_{i1}, \cdots, {}_{(1-t_{1i})}w_{iM}, R_i). \end{aligned} \tag{62}$$

In the next section we will assume separable functions which imply that spillovers only occur through the budget constraint.

3.2 Virtual prices in a Drèze equilibrium

We have assumed that the economy at each moment in time is in a Drèze equilibrium. The assumption of a Drèze equilibrium sets restrictions on how the virtual prices can vary in relationship to each other. It implies, in addition to effective demands and supplies being assumed to be Drèze demands and supplies, that the standard *min* condition applies in each market. The *min* condition says that rationing is efficient in the sense that for each micro market sellers and buyers can not be simultaneously rationed. This means, for example, that in the presence of an interior solution we can never have both $\xi_{ij}^l < (1-t_{1i})w_{ij}$ and $\eta_{ij}^l > w_{ij}$ or both $\xi_{kj}^x > p_j$ and $(1+\lambda_{f1j})\eta_j^y < p_j$ at the same time. We assume that $\xi_{ij}^l > 0$ and $\eta_{ij}^l > 0$, implying that there is always a hypothetical wage which is high enough for the worker to want to work at any firm and a hypothetical wage which is low enough for the firm to want to hire any worker. We make similar assumptions for the product markets, $\xi_{ki}^x > 0$, $\eta_{kj}^x > 0$, and $\eta_j^y > 0$.

It is important to distinguish between two situations in each market, situations where transactions are zero ($l_{ij}^s = l_{ij}^d = l_{ij}^* = 0$ in the labor markets or $y_j^s = \sum_i x_{ji}^d = 0$ in the product markets) and solutions where transactions are positive. As we shall see in the following, our assumption that each combination of firm and consumer is a separate labor market leads to a simpler classification of the labor markets than that of the product markets. The different situations which can occur in labor market ij can be characterized as follows:

I. $l_{ij}^* = 0$ occurs when one of the following are true:

- I.1 $\xi_{ij}^l > (1-t_{1i})w_{ij}$, which implies that person i is uninterested in working in firm j at wage w_{ij}
- I.2 $\eta_{ij}^l < w_{ij}$, which implies that firm j is uninterested in hiring individual i at wage w_{ij}
- I.3 the special case when we have $\eta_{ij}^l = w_{ij}$ or $\xi_{ij}^l = (1-t_{1i})w_{ij}$ at the point $l_{ij}^* = 0$, which in the first case implies that the firm is exactly indifferent to hiring or not and in the second case implies that the consumer is exactly indifferent to working or not.

The first two cases are corner solutions on one or both sides of the market, while the last is a special case of an interior solution. One should note that they also cover the case where there is a corner solution on one side of the market while the other side is rationed at the point $l_{ij}^* = 0$ (for example when the consumer is rationed, $\xi_{ij}^l < (1 - t_{1i})w_{ij}$, and the firm is uninterested in hiring, $\eta_{ij}^l < w_{ij}$).

II. $l_{ij}^* > 0$ implies an interior solution on both sides of the market (for both the individual and the firm), and will only occur in the following three cases (here we take into consideration that both the consumer and the producer can not be simultaneously rationed):

II.1 $\xi_{ij}^l = (1 - t_{1i})w_{ij}$ and $\eta_{ij}^l > w_{ij}$, the producer is rationed

II.2 $\xi_{ij}^l < (1 - t_{1i})w_{ij}$ and $\eta_{ij}^l = w_{ij}$, the consumer is rationed

II.3 $\xi_{ij}^l/(1 - t_{1i}) = \eta_{ij}^l = w_{ij}$, there is no rationing

It follows from I.3 that these are necessary but not sufficient conditions for l_{ij} to be greater than zero.

Similar conditions apply in the product markets $M_1 + 1, \ldots, M$. As mentioned, the fact that there are many agents on the buyer side of the goods markets leads to conditions which are more complicated than in the labor market where each market only consists of two agents.

III. $y_j^s = \sum_i x_{ji}^d = 0$ occurs when one of the following are true:

III.1 1 $\xi_{ji}^x < p_j$ and $\eta_{ji}^x < p_j$ for all i and j, implying that no persons or firms are interested in buying good j at price p_j.

III.2 $(1 + \lambda_{f1j})\eta_j^y > (1 - t_{2j})p_j$, which implies that firm j is uninterested in selling good j at price p_j.

III.3 The special case when we either have **(a)** $\eta_{ji}^x = p_j$ and $\xi_{ji}^x = p_j$ at the point $x_{ji}^d = 0$ for all $i \neq j$ or **(b)** $(1 + \lambda_{f1j})\eta_j^y = (1 - t_{2j})p_j$ at the point $y_j^s = 0$.

IV. $y_j^s = \sum_i x_{ji}^d > 0$ implies an interior solution on both sides of the market (for at least one buying firm or individual and for the selling firm), and will only occur in the following three cases:

IV.1 $\xi_{ji}^x = p_j$ or $\eta_{ji}^x = p_j$ for at least one i and $(1 + \lambda_{f1j})\eta_j^y < (1 - t_{2j})p_j$, the producer is rationed

IV.2 $\xi_{ji}^x > p_j$ or $\eta_{ji}^x > p_j$ for at least one i and $(1 + \lambda_{f1j})\eta_j^y = (1 - t_{2j})p_j$, the demand side is rationed

IV.3 $\xi^x_{ji} = p_j$ and $\eta^x_{ji} = p_j$ for all i and $(1 + \lambda_{f1j})\eta^y_j = (1 - t_{2j})p_j$, there is no rationing

The restrictions which apply to way the virtual prices may vary in relationship to each other when there is an interior solution as described II and IV above are the consequences of the *min* condition implicit in a Drèze equilibrium. In the markets for traded goods $1, \ldots, M_1$ the buyers are never rationed because of the possibility of importing goods. Corner solutions (no transactions in a market) are therefore solely determined from the demand side of these markets. This implies that III.1 and III.3a completely characterize the situations when $\sum_i x^d_{ji} = 0$. When there is an interior solution the relationship between sellers and buyers is not constrained by the conditions set by the Drèze equilibrium, since we apriori have assumed that only the supply side may be rationed.

Lee (1986) considers mainly the situation where there are only two agents in a closed economy and there is an interior solution in each market, making it necessary only to consider the possibilities described in II and IV above. If we allow corner solutions we also must take into account the possibilities described in I and III. The possiblity of corner solutions leads to there being a self-selection problem of the type much discusse

4 Econometric specification

We have now described an open economy non-Walrasian model with government firms and shown how the model can be formulated as an inverse supply and demand system where quantities determine virtual prices. In the following we suggest functional forms and stochastic specifications along the lines of Lee (1986) which allow estimation of the model. The suggested specification does not involve multiple integrals and is thereby more computationally tractable than general multi-market models. The following extends the set-up of Lee in allowing more than two agents, incorporating an open economy and explicitly discussing the case when there is the possibility of corner solutions.

4.1 Log-linear virtual prices

As noted in Lee (1986), for it to be feasible to estimate the above system, it is necessary to assume additive disturbances in the inverse demand and supply functions denoted by the virtual prices. The virtual prices can then be written

$$\log \xi^l_{ij}(l^s_{ij}, m^d_{ci}) = a_{ij} + \alpha_{1i} \log m^d_{ci} + \alpha_{2ij} \log (l^s_{ij} + 1) + \vartheta_{1ij}, \quad j = 1,\ldots,M, \tag{63}$$

$$\log \xi^x_{ki}(x^d_{ki}, m^d_{ci}) = c_{ki} + \alpha_{1i} \log m^d_{ci} - \alpha_{3ki} \log (x^d_{ki} + 1) + \vartheta_{3ki}, \quad k = 1,\ldots,M, \tag{64}$$

and the virtual prices for the M firms $1,\ldots,M$ in equations (46) to (48):

$$\log \eta^l_{ij}(l^d_{ij}, m^d_{fj}, K^\circ_j) \;=\; b_{ij} + \beta_{1j}\log m^d_{fj} - \beta_{2ij}\log\left(l^d_{ij}+1\right) + \beta_{3ij}\log K^\circ_j + \vartheta_{2ij}, \quad i = {}_{M+1,\ldots,M+N}, \tag{65}$$

$$\log \eta^x_{kj}(x^d_{kj}, m^d_{fj}, K^\circ_j) \;=\; c_{kj} + \beta_{1j}\log m^d_{fj} - \beta_{4kj}\log\left(x^d_{kj}+1\right) + \beta_{5kj}\log K^\circ_j + \vartheta_{3kj}, \quad k = {}_{1,\ldots,M_2}, \tag{66}$$

$$\log \eta^y_j(m^d_{fj}) \;=\; -\log\left(1+\lambda_{f1j}\right) + \beta_{1j}\log m^d_{fj} + \vartheta_{4j}, \tag{67}$$

where $\vartheta_{1ij}, \vartheta_{2ij}, \vartheta_{3ki}$, and ϑ_{4j} are stochastic variables, while the α-s, β-s, a_{ij}, b_{ij}, c_{kj}, and d_j are parameters. The stochastic variables are assumed to have white noise properties. Let $g_{1ij}(\vartheta_{1ij})$, $g_{2ij}(\vartheta_{2ij})$, $g_{3ki}(\vartheta_{3ki})$, and $g_{4j}(\vartheta_{4j})$ be the density functions of ϑ_{1ij}, ϑ_{2ij}, ϑ_{3ki}, and ϑ_{4j}, and $G_{1ij}(\vartheta_{1ij})$, $G_{2ij}(\vartheta_{2ij})$, $G_{3ki}(\vartheta_{3ki})$, and $G_{4j}(\vartheta_{4j})$ be the corresponding cumulative distribution functions. The white noise properties imply that the stochastic variables are distributed independently of each other. Labor and goods supply enter the equations in the form of $l_{ij}+1$ and $x_{ji}+1$ so as to ensure that the logarithm of the virtual prices are well-defined when transacted quantities are zero.

4.2 Behavioral interpretation

The above virtual prices can be seen as the result of utility and profit maximization under the utility and production functions

$$U_i(m_{ci}, \mathbf{x}_i, \mathbf{l}_i) \;=\; (m_{ci})^{1-\alpha_{1i}} - \sum_{j=1}^{M} a^*_{ij}(l_{ij}+1)^{1+\alpha_{2ij}} + \sum_{j=1}^{M} c^*_{ji}(x_{ji}+1)^{1-\alpha_{3ji}} \tag{68}$$

and

$$F_j(m_{fj}, \mathbf{x}_{kj}, \mathbf{l}_{ij}, K^\circ_{jk}) \;= \tag{69}$$

$$(m_{fj})^{1-\beta_{1j}} + \sum_{i=M+1}^{M+N} b^*_{ij}(l_{ij}+1)^{1-\beta_{2ij}}(K^\circ_j)^{1-\beta_{3ij}} + \sum_{i=1}^{M_2} c^*_{kj}(x_{1j}+1)^{1-\beta_{4kj}}(K^\circ_j)^{1-\beta_{5kj}},$$

where a^*_{ij}, c^*_{ji}, b^*_{ij}, and c^*_{kj} are related to a_{ij}, c_{ji}, b_{ij}, and c_{kj} in the following manner,

$$a_{ij} \;=\; \log\left[a^*_{ij}\,\frac{1+\alpha_{2ij}}{1-\alpha_{1i}}\right]$$

$$b_{ij} \;=\; \log\left[b^*_{ij}\,\frac{1+\beta_{2ij}}{1-\beta_{1j}}\right]$$

$$c_{ji} \;=\; \log\left[c^*_{ji}\,\frac{1-\alpha_{3ji}}{1-\alpha_{1i}}\right], \qquad \text{for } i = {}_{M+1,\cdots,M+N}$$

and

$$c_{kj} = \log\left[c^*_{kj}\,\frac{1-\beta_{4kj}}{1-\beta_{1j}}\right], \quad \text{for } j = 1,\cdots,M.$$

The parameters satisfy

$$\begin{array}{llll} 0<\alpha_{1i}<1, & a^*_{ij}>0, & 0<\beta_{1j}<1, & b^*_{ij}>0, \\ 0<\alpha_{2ij}, & c^*_{ji}>0, & 0<\beta_{2ij}<1, & c^*_{jk}>0, \\ 0<\alpha_{3ji}<1, & & 0<\beta_{3ij}<1, & \\ & & 0<\beta_{4kj}<1, & \\ & & 0<\beta_{5kj}<1. & \end{array}$$

The utility function is continuously differentiable, separable, increasing in x_{ji} and m_{ci}, decreasing in l_{ij} and strictly concave. The production function is continuously differentiable, increasing in all it's arguments, separable, and strictly concave. The assumption of separability implies that spillovers only occur indirectly through the budget constraint. The above specification of the production function implies decreasing returns to scale if $\beta_{2ij}+\beta_{3ij}>1$ and $\beta_{4kj}+\beta_{5kj}>1$. The capital stock's impact on the productivity of labor is reflected in the parameter β_{3ij} and on the productivity of other inputs in the parameter β_{5kj}.

The quasi-concavity of the utility function is, as mentioned earlier, problematic when the consumer has so many job possibilities, since in practice most individuals only have one or at most two jobs. The above specifications of the utility and production functions have the drawback that it is not straightforward to derive ordinary demand and supply functions for the consumers and firms. The above specification of the production function leads to the following marginal rates of substitution[2],

$$\frac{\partial F_j/\partial l_{ij}}{\partial F_j/\partial K^\circ_j} = \frac{1-\beta_{2ij}}{1-\beta_{3ij}}\,\frac{(l_{ij}+1)^{-\beta_{2ij}}\,K^\circ_j}{1+\frac{c^*_{jk}(1-\beta_{5kj})}{b^*_{ij}(1-\beta_{3ij})}\,(K^\circ_j)^{(\beta_{3ij}-\beta_{5kj})}}, \tag{70}$$

$$\frac{\partial F_j/\partial x_{kj}}{\partial F_j/\partial K^\circ_j} = \frac{1-\beta_{4kj}}{1-\beta_{5kj}}\,\frac{(x_{kj}+1)^{-\beta_{4kj}}\,K^\circ_j}{1+\frac{b^*_{ij}(1-\beta_{3ij})}{c^*_{kj}(1-\beta_{5kj})}\,(K^\circ_j)^{(\beta_{5kj}-\beta_{3ij})}}, \tag{71}$$

and

$$\frac{\partial F_j/\partial l_{ij}}{\partial F_j/\partial x_{kj}} = \frac{b^*_{ij}(1-\beta_{2ij})}{c^*_{kj}(1-\beta_{4kj})}\,\frac{(l_{ij}+1)^{-\beta_{2ij}}(K^\circ_j)^{(\beta_{5kj}-\beta_{3ij})}}{(x_{kj}+1)^{-\beta_{4kj}}}. \tag{72}$$

[2]Expressions for the marginal rate of substitution of capital are included, even though capital is exogenous in our model.

We see that the above marginal rates of substitution are simplified if $\beta_{3ij} = \beta_{5kj}$. The above functional forms imply that in the case of the consumer the relaxation of a quantity constraint for a good will reduce the demand for other goods and increase the supply of labor, while the relaxation of a quantity constraint for labor supplied to a certain firm will increase the demand for other goods and decrease the supply of labor to other firms. The same type of implications hold for the firm.

The above specifications of the utility and production functions leads to a more general specification of the virtual prices than one gets from the generalized Cobb-Douglas utility and production functions

$$U_i = \log m_{ci} + \sum_{1}^{M} a_{ij}^* \log(\bar{L} - l_{ij}) + \sum_{k=1}^{M} c_{ki}^* \log x_{ki} \tag{73}$$

and

$$F_j = \log m_{fj} + \sum_{1}^{M} b_{ij}^* \log(l_{ij} + 1) + \sum_{k=1}^{M} c_{ki}^* \log x_{kj}. \tag{74}$$

These functional forms also lead to log-linear virtual prices but with less parameters than those in equations (68) and (69). For example the logarithm of consumer i's virtual price for labor supplied to firm j will be

$$\log \xi_{ij}^l = \log a_{ij} + \log \frac{m_{ci}}{\bar{L} - l_{ij}}. \tag{75}$$

The log-linear virtual price utility and production functions described in equations (68) and (69) were chosen because they give a richer parameterization of the virtual prices than the generalized Cobb-Douglas function.

4.3 Possible regimes and the likelihood function

The model has M product markets and $M \cdot N$ labor markets. In the labor markets and the $M_1 + 1, \ldots, M$ markets for non-traded and government goods either the demand side or the supply side is rationed (with the probability of no rationing being of measure zero). Assuming interior solutions in these markets leads, as discussed in the preceeding section, to there being two possible situations in each market. In the labor market either situation II.1 or II.2 from section 3 will apply, and in the product markets $M_1 + 1, \ldots, M$ either IV.1 or IV.2 will apply. We define a regime as one possible combination of such rationing situations in these markets . In total there will be $2^{MN+(M-M_1)}$ different mutually exclusive regimes (except for overlaps which have probabilities of measure zero). Regime k will consist of a vector of $MN + (M - M_1)$ elements each describing the rationing situation i one of the markets.

We now introduce a variable $J_{ij}(k)$ describing the rationing situation in labor market ij under regime k and a variable $I_j(k)$ describing the rationing situation in commodity market j under regime k. For notational simplicity we let $\lambda_j = 1 - \lambda_{f1j}$. The two variables are defined as

$$J_{ij}(k) \quad = \quad \begin{cases} 1 & \text{if } \eta^l_{ij} \geq (1-t_{1i})w_{ij} \text{ and } \xi^l_{ij} = w_{ij} \\ 0 & \text{if } \eta^l_{ij} = w_{ij} \text{ and } \xi^l_{ij} \leq (1-t_{1i})w_{ij} \end{cases} \tag{76}$$

for all i and j,

$$I_j(k) \quad = \quad \begin{cases} 1 & \text{if } \lambda_j \eta^y_j \leq (1-t_{2j})p_j, \\ & \quad \eta^x_{jk} = p_j \text{ for all } k \neq j, k = {}_{1,\ldots,M}, \\ & \quad \text{and } \xi^x_{ji} = p_j \text{ for all } i = {}_{M+1,\ldots,M+N}, \\ 0 & \text{if } \lambda_j \eta^y_j = (1-t_{2j})p_j \\ & \quad \eta^x_{jk} \geq p_j \text{ for all } k \neq j, k = {}_{1,\ldots,M}, \\ & \quad \text{and } \xi^x_{ji} \geq p_j \text{ for all } i = {}_{M+1,\ldots,M+N}, \end{cases} \tag{77}$$

for $j = {}_{M_1+1,\ldots,M_2}$, and

$$I_j(k) \quad = \quad \begin{cases} 1 & \text{if } \lambda_j \eta^y_j \leq (1-t_{2j})p_j \text{ and } \xi^x_{ji} = p_j \text{ for all } i = {}_{M+1,\ldots,M+N}, \\ 0 & \text{if } \lambda_j \eta^y_j = (1-t_{2j})p_j \text{ and } \xi^x_{ji} \geq p_j \text{ for all } i = {}_{M+1,\ldots,M+N}, \end{cases} \tag{78}$$

for $j = {}_{M_2+1,\ldots,M}$. The vector

$$[I_{M_1+1}(k), \ldots, I_M(k), J_{11}(k), \ldots, J_{1M}(k), , \ldots, \ldots, J_{M+N\,1}(k), \ldots, J_{M+N\,M}(k)]$$

will then constitute a description of regime k. Each regime implies that a subset of the virtual prices in equations (63) to (67) can be set equal to observed prices or wages. The equations which apply in regime k will be

$$\left.\begin{array}{lll} \log\left((1-t_{1i})w_{ij}\right) = a_{ij} + \alpha_{1i}\log m^d_{ci} + \alpha_{2ij}\log\left(l^s_{ij}+1\right) + \vartheta_{1ij} & \text{if} & J_{ij}(k) = 1 \\ \log\left((1-t_{1i})w_{ij}\right) = b_{ij} + \beta_{1j}\log m^d_{fj} - \beta_{2ij}\log\left(l^d_{ij}+1\right) & \text{if} & J_{ij}(k) = 0 \\ \qquad\qquad +\beta_{3ij}\log K^\circ_j + \vartheta_{2ij} & & \end{array}\right\} \quad \begin{array}{l} i = {}_{M+1,\ldots,M+N} \\ j = {}_{1,\ldots,M} \end{array}$$

$$
\left.
\begin{aligned}
\log p_j &= c_{j1} + \beta_{11}\log m^d_{f1} - \beta_{4j1}\log\left(x^d_{j1}+1\right) \\
&\quad +\beta_{5j1}\log K^\circ_1 + \vartheta_{3j1} \\
&\quad\quad \vdots \\
\log p_j &= c_{jM} + \beta_{1M}\log m^d_{fM} - \beta_{4jM}\log\left(x^d_{jM}+1\right) \\
&\quad +\beta_{5jM}\log K^\circ_M + \vartheta_{3jM} \\
\log p_j &= c_{j\,M+1} + \alpha_{1\,M+1}\log m^d_{c\,M+1} \\
&\quad -\alpha_{3j\,M+1}\log\left(x^d_{j\,M+1}+1\right) + \vartheta_{3j\,M+1} \\
&\quad\quad \vdots \\
\log p_j &= c_{j\,M+N} + \alpha_{1\,M+N}\log m^d_{c\,M+N} \\
&\quad -\alpha_{3j\,M+N}\log\left(x^d_{j\,M+N}+1\right) + \vartheta_{3j\,M+N} \\
\log\left((1-t_{2j})p_j\right) &\le -\log\lambda_j + \beta_{1j}\log m^d_{fj} + \vartheta_{4j}
\end{aligned}
\right\} \quad j = {}_{1,\ldots,M_1}
$$

$$
\left.
\begin{aligned}
&\left.
\begin{aligned}
\log p_j &= c_{j1} + \beta_{11}\log m^d_{f1} - \beta_{4j1}\log\left(x^d_{j1}+1\right) \\
&\quad +\beta_{5j1}\log K^\circ_1 + \vartheta_{3j1} \\
&\quad\quad \vdots \\
\log p_j &= c_{jM} + \beta_{1M}\log m^d_{fM} - \beta_{4jM}\log\left(x^d_{jM}+1\right) \\
&\quad +\beta_{5jM}\log K^\circ_M + \vartheta_{3jM} \\
\log p_j &= c_{j\,M+1} + \alpha_{1\,M+1}\log m^d_{c\,M+1} \\
&\quad -\alpha_{3j\,M+1}\log\left(x^d_{j\,M+1}+1\right) + \vartheta_{3j\,M+1} \\
&\quad\quad \vdots \\
\log p_j &= c_{j\,M+N} + \alpha_{1\,M+N}\log m^d_{c\,M+N} \\
&\quad -\alpha_{3j\,M+N}\log\left(x^d_{j\,M+N}+1\right) + \vartheta_{3j\,M+N}
\end{aligned}
\right\} \quad \text{if} \quad I_j(k) = 0 \\
&\log\left((1-t_{2j})p_j\right) = -\log\lambda_j + \beta_{1j}\log m^d_{fj} + \vartheta_{4j} \qquad \text{if} \quad I_j(k) = 1
\end{aligned}
\right\} \quad j = {}_{M_1+1,\ldots,M_2}
$$

$$
\left.
\begin{array}{ll}
\log p_j = c_{j\,M+1} + \alpha_{1\,M+1} \log m^d_{c\,M+1} & \\
\qquad\qquad -\alpha_{3j\,M+1} \log\left(x^d_{j\,M+1} + 1\right) + \vartheta_{3j\,M+1} & \\
\qquad\qquad \vdots & \text{if} \quad I_j(k) = 0 \\
\log p_j = c_{j\,M+N} + \alpha_{1\,M+N} \log m^d_{c\,M+N} & \\
\qquad\qquad -\alpha_{3j\,M+N} \log\left(x^d_{j\,M+N} + 1\right) + \vartheta_{3j\,M+N} & \\
\\
\log\left((1 - t_{2j}) p_j\right) = -\log \lambda_j + \beta_{1j} \log m^d_{fj} + \vartheta_{4j} & \text{if} \quad I_j(k) = 1
\end{array}
\right\} \quad j = {}_{M_2+1,\ldots,M}
$$

The likelihood function is based on the distribution of the observed variables transformed from the stochastic distribution assumed for the stochastic variables. This transformation is based on the above system of equations. Our definition of regime discribes only situations involving interior solutions, while one must often in addition take the possibility of corner solutions into account. The joint probability density of the two stochastic variables involved in labor market ij is given by the variable P^l_{1ij} when the supply side is rationed ($J_{ij}(k) = 1$) and by the variable P^l_{2ij} when the demand side is rationed ($J_{ij}(k) = 0$). These two probability densities can be written

$$
\begin{aligned}
\mathrm{P}^l_{1ij} &= \Pr\left(\vartheta_{1ij}, \vartheta_{2ij} \mid \eta^l_{ij} = w_{ij}, \xi^l_{ij} \leq (1 - t_{1i}) w_{ij}\right) \\
&= g_{2ij}(\vartheta_{2ij}) \cdot G_{1ij}(\vartheta_{1ij}) \cdot \mathrm{P}^*_{ij} \qquad (79)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{P}^l_{2ij} &= \Pr\left(\vartheta_{1ij}, \vartheta_{2ij} \mid \eta^l_{ij} \geq w_{ij}, \xi^l_{ij} = (1 - t_{1i}) w_{ij}\right) \\
&= g_{1ij}(\vartheta_{1ij}) \cdot \left[1 - G_{2ij}(\vartheta_{2ij})\right] \cdot \mathrm{P}^*_{ij} \qquad (80)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{P}^*_{ij} &= \Pr\left(l^d_{ij} > 0, l^s_{ij} > 0\right) \\
&= \left[1 - G_{1ij}(\vartheta_{1ij})\right] + G_{2ij}(\vartheta_{2ij}) - \left[1 - G_{1ij}(\vartheta_{1ij})\right] G_{2ij}(\vartheta_{2ij}) \qquad (81)
\end{aligned}
$$

for $i = M+1, \ldots, M+N$. The probabilities above consist of a density for the disturbance term of the equation for the side which is not rationed (for example in the last equation this will be $g_{1ij}(\vartheta_{1ij})$) times the probability that the other side is rationed ($[1 - G_{2ij}(\vartheta_{2ij})]$) times the probability that there is no corner solution in the market (P^*_{ij}). The probability of corner solutions is explicitly taken into account through the probability P^*_{ij}, which gives the probability that there is an interior solution in labor market ij. The probability P^*_{ij} can be viewed as a sample selection correction of the type often employed in econometric analysis of labor supply.

Similar probability densities can be derived for the goods markets $j = M_1 + 1, \ldots, M$. The joint probability density of the stochastic variables (assuming that they are independtly distributed) in goods market j is given by the variable P^x_{1j} when the supply side is rationed ($I_j(k) = 1$) and by the variable P^x_{2j} when the demand side is rationed ($I_j(k) = 0$). These two probability densities can be written

$$\begin{aligned}
\mathrm{P}^x_{1j} &= \Pr\big(\vartheta_{4j}, \vartheta_{3j1}, \vartheta_{3j2}, \ldots \vartheta_{3j\,M+N} \mid \lambda_j \eta^y_j \leq {}_{(1-t_{2j})} p_j,\ \eta^x_{jk} = p_j \\
&\qquad \text{or } \xi^x_{jk} = p_j \text{ for at least one } k \neq j\big) \\
&= G_{4j}(\vartheta_{4j}) \cdot \prod_{k \neq j} \big(g_{3jk}(\vartheta_{3jk}) \cdot \Pr\big(x^d_{jk} > 0 \mid x^d_{jk} > 0 \text{ for all at least one } k\big)\big) \\
&\qquad \cdot \Pr\big(y^s_j > 0,\ x^d_{jk} > 0 \text{ for at least one } k\big) \\
&= G_{4j}(\vartheta_{4j}) \cdot \prod_{k \neq j} g_{3jk}(\vartheta_{3jk}) \cdot \mathrm{P}^{**}_j \cdot \prod_{k \neq j} \frac{G_{3kj}(\vartheta_{3kj})}{1 - \prod_{k \neq j} G_{3kj}(\vartheta_{3kj})} \qquad (82)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{P}^x_{2j} &= \Pr\big(\vartheta_{3j1}, \vartheta_{3j2}, \ldots \vartheta_{3j\,M+N} \mid \lambda_j \eta^y_j = {}_{(1-t_{2j})} p_j,\ \eta^x_{jk} \geq p_j \\
&\qquad \text{or } \xi^x_{ji} \geq p_j \text{ for at least one } i = {}_{1,\ldots,M+N}\big) \\
&= g_{4j}(\vartheta_{4j}) \cdot \prod_{k \neq j} [1 - G_{3jk}(\vartheta_{3jk})] \cdot \mathrm{P}^{**}_j, \qquad (83)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{P}^{**}_j &= \Pr\big(y_j > 0,\ x^d_{jk} > 0 \text{ for at least one } k,\big) \qquad (84) \\
&= [1 - G_{4j}(\vartheta_{4j})] + [1 - \prod_{k \neq j} G_{3kj}(\vartheta_{3kj})] - [1 - G_{4j}(\vartheta_{4j})][1 - \prod_{k \neq j} G_{3kj}(\vartheta_{3kj})]
\end{aligned}$$

for $j = M_1 + 1, \ldots, M$. If we assume that there is always is an interior solution we can set $P^*_{ij} = 1$ and $P^{**}_j = 1$. For the $1, \ldots, M_1$ markets where the demand side never is rationed we have the following probability density for the stochastic variables:

$$\begin{aligned}
P^x_{0j} &= \Pr\left(\vartheta_{4j}, \vartheta_{3j1}, \vartheta_{3j2}, \ldots \vartheta_{3j\,M+N}\right) \\
&= g_{4j}(\vartheta_{4j}) \cdot \prod_{k \neq j} g_{3jk}(\vartheta_{3jk}) \cdot [1 - G_{4j}(\vartheta_{4j})] \cdot \prod_{k \neq j} G_{3jk}(\vartheta_{3jk}) \qquad (85)
\end{aligned}$$

where $[1 - G_{4j}(\vartheta_{4j})] \cdot \prod_{k \neq j} G_{3jk}(\vartheta_{3jk})$ is the probability of an interior solution.

We see from the above that these probability densities have a simpler structure for the labor market than for the goods markets. Using the above notation we can now write regime k's contribution to the likelihood function as

$$\mathcal{L}_k(\alpha,\beta,a,b,c,\lambda,\theta \mid \mathbf{m}^d,\mathbf{l}^*,\mathbf{x}^d) = |H_k(\mathbf{m}^d,\mathbf{l}^*,\mathbf{x}^d)| \prod_{j=1}^{M_1} P_{0j}^x \prod_{j=M_1+1}^{M} (P_{1j}^x)^{I_j(k)}$$

$$\cdot \prod_{j=M_1+1}^{M} (P_{2j}^x)^{((1-I_j(k))} \prod_{j=M+1}^{M+N} (P_{1ij}^l)^{J_{ij}(k)} \prod_{j=M+1}^{M+N} (P_{2ij}^l)^{(1-J_{ij}(k))} \tag{86}$$

where $H_k(\mathbf{m}^d,\mathbf{l}^*,\mathbf{x}^d)$ is the Jacobian of the transformation from the ϑ-s to the observed variables in regime k, $\alpha,\beta,a,b,c,\lambda$ are vectors of the structural parameters, and θ is a vector of parameters of the distribution functions. The regimes are defined so that they are mutually exclusive (with exception for situations with probability of measure zero). The total likelihood function will be

$$\mathcal{L}(\alpha,\beta,a,b,c,\lambda,\theta \mid \mathbf{m}^d,\mathbf{l}^*,\mathbf{x}^d) = \prod_k \mathcal{L}_k(\alpha,\beta,a,b,c,\lambda,\theta \mid \mathbf{m}^d,\mathbf{l}^*,\mathbf{x}^d). \tag{87}$$

The likelihood does not involve multiple integrals, but may still be computationally cumbersome when there are many markets, because increases in the number of markets increases the number of regimes exponentially. In addition, the more markets there are, the more important it becomes to take into consideration corner solutions, and it might become difficult to get detailed enough micro data. It would seem that at present one in empirical disequilibrium work is restricted to either work within a representative agent framework assuming a small number of markets (but more than the usual two markets) or devise methods for aggregating across a large number of markets.

5 Summary

This paper has discussed a multi-market non-Walrasian model with many agents which can be used for empirical work when there are a large number of markets. The main aim has been to develop a method for describing the extent of rationing in an economy and to estimate structural parameters under rationing. Rationing in the economy is in the model revealed implicitly through the difference between the observed transactions and the transactions that are optimal for the agents. The framework used in the paper is mainly an extension of the virtual price approach suggested by Lee (1986), allowing for more than two agents, incorporating an open economy and explicitly taking into account the possibility of corner solutions. It is assumed that exports, the capital stock, investment, and the budget constraints of the government firms are exogenous in the model, while imports, the trade surplus, tax revenue, the public budget deficit, and changes in

the money supply are endogenous. The model is based on explicit utility and profit maximization, where both consumers and firms face budget constraints. These constraints and the introduction of money into the utility and profit functions can be considered as ways of introducing liquidity constraints. The rather primitive nature of these assumptions are partly related to the static nature of the model. Any future extension of the model to include for example price determination will introduce dynamic considerations making it desirable to also look closer at how expectations are formed and at the intertemporal allocation of assets.

In the model it was assumed that the economy at any time was in a Drèze equilibrium and it was demonstrated that such an equilibrium exists. The modeling of the labor markets was built on the assumption that each combination of worker and firm was a separate micro labor market, leading to a simplified econometric modeling of the labor market. The model was formulated as an inverse supply and demand system where observed quantities determined virtual prices. The assumption of a Drèze equilibrium sets restrictions on how the virtual prices could vary in relationship with each other. These restrictions were utilized in caluculating the likelihood function for the observed variables.

The econometric specification of the model assumed that the virtual prices were log-linear in the observed quantities with additive random variables. The parameters can be interpreted as structural parameters derived from the agents' utility and production functions. The chosen specification is such that the likelihood does not involve multiple integrals, as originally shown in Lee (1986).

Despite the use of a simple econometric specification it is apparent that when there are a large number of markets, the computational burden of estimation becomes heavy due to the large number of possible rationing regimes. Increases in the number of markets increases the number of regimes exponentially. In addition, the more markets there are, the more important it becomes to take into consideration corner solutions, and it becomes more difficult to get detailed enough micro data. It therefore seems that in econometric disequilibrium work one is either restricted to working with representative agent models such as the examples discussed at the end of the paper or one must derive explicit aggregate relationships. A problem with representative agent models is that the number of individuals or firms represented by each agent will vary over time. In a later paper the framework discussed in this paper will be the point of departure for deriving explicit aggregate labor market relationships.

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A Existence of Drèze equilibrium

In the following we will prove the existence of a Drèze equilibrium for the model discussed earlier in the paper. The proof is a variant of that first presented in Drèze (1975). The exposition borrows heavily from Mukherji's (1990) pp. 153-157 version of the original proof in Drèze (1975). Before proceeding to the proof, some added notation is introduce so that it is possible to represent prices, quantities (demanded, supplied and transacted quantities), and constraints for all goods and labor in a parsimonious manner. Our assumption that each combination of firm and individual is a separate type of labor leads to the notation being a bit complex. The firms' maximization problem is formulated in a manner which is analogous to the consumers' (maximizing a criterion function, which is strictly quasi-concave in the decision variables, subject to a budget constraint). The first part of the proof (sections A.1 and A.2) proves the continuity of the consumers' and the firms' Drèze demand and supply functions. These proofs are based on first showing that the budget correspondences faced by the consumers and those faced by the firms are lower hemicontinuous (lemma 1 and lemma 3). The lower hemicontinuity proofs for the consumers and firms differ in that in the case of the consumers there is a constraint on hours worked, while in the case of the firms the production function induces a constraint on their output. After showing lower hemicontinuity, continuity of the Drèze demand and supply functions is proved (lemma 2 and lemma 4). Finally, in section A.3 the existence of maximum and minimum constraints for any set of prices and exogenous incomes is proved (theorem 1), under the condition that demands and supplies are of the Drèze type (condition 1.2 in theorem 1) and that the *min* condition is satisfied for each good and for each type of labor (condition 1.3 in theorem 1). The proof of theorem 1 is based on using Brouwer's Fixed Point Theorem.

As mentioned above, we need to introduce some new notation in addition to that used in the main part of the paper. This notation is based on vectors of all the goods and labor in the model instead of just those pertaining to each agent. Such notation makes it easier to formulate the equilibrium conditions. From the main part of the paper we had that the $1+M+MN$ vector of commodities and labor exchanged in the economy was

$$[m, x_1, \ldots, x_M, l_{M+1}, \ldots, l_{M+N}]$$

where the vector $l_i = [l_{i1}, l_{i2}, \ldots, l_{iM}]$ is individual i's supply of labor, m is money, and x_i is the commodity produced by firm i. The number of firms is M and the number of individuals is N. As discussed earlier in the paper each individual's potential supply of labor to each firm is viewed as a separate commodity. This leads to there being $M \cdot N$ types of labor and $M+1$ types of goods

(including money). The corresponding price/wage vector q can be written

$$q \quad = \quad [1, p_1, \ldots, p_M, w_{M+1}, \ldots, w_{M+N}].$$

where the vector $w_i = [w_{i1}, w_{i2}, \ldots, w_{iM}]$ contains the wages which apply to individual i's supply of labor. The elements of the vector q are numbered from 0 to $M+MN$ so that we have $q_0 = 1$, $q_1 = p_1$ and so forth.

We index the agents, as before, both firms and individuals by $i = 1,\ldots,M+N$, where $1,\ldots,M$ are firms and $i = M+1,\ldots,M+N$ are consumers. Their net demand of commodities and labor is given by the $(1+M+MN)$ vector z^i. In the case of the consumers ($i = M+1,\ldots,M+N$) money is denoted by $z^i_0 = m_{ci}$, commodities with disutility (labor supply) by $z^i_j = -l_{ik}$,[3] and the remaining commodities by $z^i_j = x_{ji}$. The consumption/labor supply vector for consumer i can thereby be written

$$\begin{aligned} z^i &= [z^i_0, \ldots, z^i_n] \\ &= [m_{ci}, x_{1i}, \ldots, x_{Mi}, 0, \ldots, 0, -l_{i1}, -l_{i2}, \ldots, -l_{iM}, 0, \ldots, 0], \end{aligned}$$

where $n = M+MN$.

In the case of the firms we denote money by $z^i_0 = m_{fi}$, labor demand by $z^i_j = l_{ki}$,[4] and the remaining inputs by $z^i_j = x_{ji}$. It simplifies the notation later to exclude the firm's output from the vector z^i by setting $z^i_i = 0$. As before we denote firm i's endogenous output by y^i and the exogenous output (exports and production for investment purposes) by Y^i. The input vector for firm i can thereby be written

$$\begin{aligned} z^i &= [z^i_0, \ldots, z^i_n] \\ &= [m_{fi}, x_{1i}, \ldots, x_{(i-1)i}, 0, x_{(i+1)i}, \ldots, x_{M_2 i}, \\ &\quad 0, \ldots, 0, l_{M+i}, 0, \ldots, 0, l_{2M+i}, 0, \ \ldots \ , 0, l_{MN+i}, 0, \ldots, 0]. \end{aligned}$$

The taxes faced by consumer i are given by the $1+M+MN$ vector τ^i,

$$\tau^i \quad = \quad [1, \ldots, 1, (1-t_{1i}), \ldots, (1-t_{1i}), 1, \ldots, 1] \qquad \text{for } i = M+1,\ldots,M+N,$$

where $(1-t_{1i}), \ldots, (1-t_{1i})$ are the taxes on labor income paid by worker i. Since such taxes do not vary between firms this series consists of M equal tax rates.

The maximum and minimum constraints faced by the consumers ($i = M+1,\ldots,M+N$) are denoted by

$$S^i \quad = \quad [S^i_{M_1+1}, \ldots, S^i_M] \ = \ [\overline{x}_{M_1+1\,i}, \ldots, \overline{x}_{Mi}]$$

[3] The relationship between the subscripts is given by $j = (i-M)M + k$ or $k = j - (i-M)i$.

[4] The relationship between the subscripts is given by $j = (k-M)\cdot M + i$ or $k = (j-i)/M + M$.

and

$$s^i \;=\; [s^i_{(i-M)M+1},\ldots,s^i_{(i-M)M+M}] \;=\; [\underline{l}_{i1},\ldots,\underline{l}_{iM}],$$

where $S^i_j = \overline{x}_{ji} \geq 0$ and $s^i_j = \underline{l}_{ik} \leq 0$.[5] The vector S^i denotes the constraints on the amount of goods that can be bought from the firms producing non-tradeable goods, while the vector s^i denotes the constraints on the amount of labor consumer i can supply to these firms.

In the same manner the constraints faced by the firms ($i = 1,\ldots,M$) are denoted

$$s^i \;=\; [s^i_i] \;=\; \underline{y}_i$$

and

$$\begin{aligned} S^i &= [S^i_{M_1+1},\ldots,S^i_{i-1},S^i_{i+1},S^i_{M_2},S^i_{M+i},S^i_{2M+i},S^i_{3M+i},\ldots,S^i_{MN+i}] \\ &= [\overline{x}_{M_1+1\,i},\ldots,\overline{x}_{1\,i-1},\overline{x}_{1\,i+1},\ldots,\overline{x}_{M_2 i},\overline{l}_{M+1\,i},\ldots,\overline{l}_{M+N\,i}], \end{aligned}$$

where $S^i_j = \overline{x}_{ji} \geq 0$ are constraints on the input of commodities when $j = M_1+1,\ldots,M$ and $S^i_j = \overline{l}_{ki} \geq 0$ are constraints on labor when $j = M+i,2M+i,3M+i,\ldots,MN+i$.[6] Note that the constraint $s^i \leq 0$ is not a constraint on the production possibility set, but a constraint on sales. There is no constraint on money, z^i_0.

Using the above notation the utility function u^i for consumer i can be written

$$u^i(z^i) \;=\; U_i(m_{ci},\mathbf{x}_i,\mathbf{l}_i), \tag{A.1}$$

where U_i is the utility function used in the earlier part of the paper. The utility function u^i is an increasing function of all its arguments. The production function f^i can be written as

$$f^i(z^i) = F_i(m_{fi},\mathbf{x}_i,\mathbf{l}_i,K^i), \tag{A.2}$$

where F_i is the concave production function from the main part of the paper and K^i is the capital stock at the beginning of the period. The production and utility functions are assumed to be continuously differentiable functions, the utility function to be strictly quasi-concave in the consumers' relevant variables, and the production function to be strictly concave in the firms' relevant variables[7] (by relevant variable we mean a variable which level can be chosen by the agent in question, see also footnote 9). We also assume that $f(\cdot)$ and thereby $f^i(z^i)$ is non-singular so that $f^i(z^i) = 0$ implies $z^i = 0$.

[5] See footnote 2.

[6] See footnote 3.

[7] Even though the functions $u^i(z_i)$ is not *strictly* quasi-concave or $f^i(z_i)$ *strictly* concave in all the arguments, the fact that they are so in all variables relevant to the agent ensures that we get a unique solution to each agents maximization problem.

The profit function for firm i, π^i, is defined as[8]

$$\pi^i(z^i) \quad = \quad (1-t_{2i})q_i y^i - qz^i. \tag{A.3}$$

The continuity and differentiability of the production function imply that the profit function π^i is also continuous and differentiable.

Note that the profit function $\pi^i(z^i)$ is strictly quasi-concave in z^i if the production function $f^i(z^i)$ is strictly concave. This is easily seen by assuming two sets of production inputs z and z' (the index i is suppressed) such that $\pi(z) \geq \pi(z')$. $\pi(z) \geq \pi(z')$ implies that $f(z) - f(z') \geq \frac{q}{p}(z - z')$. Strict concavity of $f(z)$ implies that $f(\lambda z + (1-\lambda)z') > \lambda f(z) + (1-\lambda)f(z')$ for $0 \leq \lambda \leq 1$. Taking these two equations together leads to $f(\lambda z + (1-\lambda)z') - f(z') > \lambda\frac{q}{p}(z - z')$ implying that $\pi(\lambda z + (1-\lambda)z') > \pi(z')$ which is the definition of strict quasi-concavity.

Above we have defined the vector z^i so as to include all goods and types of labor in the economy. Since each type of labor is firm and individual specific, many of the elements in this vector must be zero. Individual i cannot for example sell individual j's labor. We let the set $\mathcal{A}^i$ denote the elements of the vector z^i which are independent choice variables for agent i (all other variables will be zero)[9]. For consumer i it will contain $1+2M$ elements, $\mathcal{A}^i = \{z_0^i, z_1^i, \ldots, z_M^i, z_{M+M(i-M-1)+1}^i, \ldots, z_{M+M(i-M-1)+M}^i\}$, while for firm i it will contain $N+M$ elements, $\mathcal{A}^i = \{z_0, z_1^i, \ldots, z_{i-1}^i, z_{i+1}^i, \ldots, z_{M_2}^i, z_{M+i}^i, z_{2M+i}^i, z_{3M+i}^i, \ldots, z_{MN+i}^i\}$. The firm's output, z_i^i, is not included, since it follows from the production function when the inputs have been chosen. Let C^i denote the initial quantity money of agent i at the beginning of the period and $\bar{L}$ the maximum time each consumer can work. Now consider the utility and profit maximization problems

$$\begin{aligned} \max_{z^i \in \mathcal{A}^i} \quad & u^i(z^i) \\ \text{s.t.} \quad & z^i \in \gamma_c^i(s^i, S^i, q, \tau^i, C^i, \bar{L}) \end{aligned}$$

and

$$\begin{aligned} \max_{z^i \in \mathcal{A}^i} \quad & \pi^i(z^i) \\ \text{s.t.} \quad & z^i \in \gamma_f^i(s^i, S^i, q, \tau^i, C^i, Y^i) \end{aligned}$$

[8] Note that we use π for profits as a function of the inputs z and not in the usual sense of being the maximal profits as a function of prices.

[9] Given agent i's constraint set α_i, the variable z_j^i is a *relevant variable* if there exists a $\hat{z}_j^i$ in α_i such that $\hat{z}_j^i > 0$. The set $\mathcal{A}^i$ is then the set of relevant variables for agent i.

where

$$\begin{aligned}\gamma_c^i(s^i, S^i, q, \tau^i, C^i, \bar{L}) \quad = \quad & \big\{z^i : \tau^i q z^i \leq C^i, \\ & \sum_{j=M+1}^{n} z_j^i \leq \bar{L}, \\ & 0 \leq z_j^i, \quad j = {}_{M_1+1,\ldots,M_1}, \\ & 0 \leq z_j^i \leq S_j^i, \quad j = {}_{M_1+1,\ldots,M}, \\ & s_j^i \leq -z_j^i \leq 0, \quad j = {}_{(i-M)M+1,\ldots,(i-M)M+M}, \\ & z_0^i \geq 0, \\ & z_j^i = 0, \quad \text{for all } \; j \notin \mathcal{A}^i \big\}\end{aligned}$$

and

$$\begin{aligned}\gamma_f^i(s^i, S^i, q, \tau^i, C^i, Y^i) \quad = \quad & \big\{z^i : q z^i \leq C^i, \\ & s_i^i \leq -f^i(z^i) + Y^i \leq 0, \\ & 0 \leq z_j^i, \quad j = {}_{1,\ldots,M_1}, \;\; j \neq i, \\ & 0 \leq z_j^i \leq S_j^i, \quad j = {}_{M_1+1,\ldots,M_2}, \;\; j \neq i, \\ & 0 \leq z_j^i \leq S_j^i, \quad j = {}_{M+i,2M+i,3M+i,\ldots,MN+i}, \\ & z_0^i \geq 0, \\ & z_j^i = 0, \quad \text{for all } \; j \notin \mathcal{A}^i \big\}.\end{aligned}$$

Utility maximization yields the vector of Drèze demands and supplies $z^i(s^i, S^i, q, \tau^i, C^i, \bar{L})$ for each consumer, while profit maximization yields the vector of Drèze demands $z^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$ for each firm. The strict quasi-concavity of the utility and profit functions in the relevant variables assures that the utility and profit maximization problems have unique solutions. We set the element covering the firm's output equal to zero, $z_i^i(s^i, S^i, q, t_{2i}, C^i, Y^i) = 0$, and instead let the Drèze output from firm i be denoted $y^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$. All elements of these demand vectors which are not choice variables for the agent are equal to zero. The only difference between private and government firms in the present context is that for government firms the sales constraint is an equality, $f^i(z^i) = -s_i^i$.

Before proceeding to the proof we define what is meant by the budget set and the budget correpondence of the agents.

Definition *Let γ_c^i be a correspondence $R_-^M \times R_+^M \times R_+^{n+1} \times R_+^n \times R_+ \times R_+ \to R_+^{n+1}$ such that $z^i \in \gamma_c^i(s^i, S^i, q, \tau^i, C^i, \bar{L})$. The function γ_c^i will be referred to as the consumer's budget correspondence and $\gamma_{c^i}(s^i, S^i, q, \tau^i, C^i, \bar{L})$ as her budget set. In the same manner we define the firm's budget*

correspondence γ_f^i *as the correspondance* $R_- \times R_+^{M+N} \times R_+^{n+1} \times R_+^n \times R_+ \times R_+ \to R_+^{n+1}$ *such that* $z^i \in \gamma_f^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$*, where* $\gamma_f^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$ *is the firm's budget set.*

A.1 Continuity of the consumers' Drèze demand and supply functions

Lemma 1 *The consumer's budget correspondence* γ_c^i *is lower hemicontinuous at every point* $(s^i, S^i, q, \tau^i, C^i, \bar{L})$ *such that* $q_j > 0$*,* $\tau_j^i > 0$*,* $s_j^i \leq 0 \leq S_j^i$ *for all* j*,* $C^i > 0$*, and* $\bar{L} > 0$*.*

Proof: We prove this in three steps. Before proceeding we define

$$\begin{aligned}
\alpha_c^i(s^i, S^i) \quad = \quad & \{z^i : 0 \leq z_j^i, \quad j = 1,\ldots,M_1, \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M_1+1,\ldots,M, \\
& s_j^i \leq -z_j^i \leq 0, \quad j = (i-M)M+1,\ldots,(i-M)M+M, \\
& z_0^i \geq 0, \\
& z_j^i = 0, \quad \text{for all } \; j \notin \mathcal{A}^i\}
\end{aligned}$$

and

$$\begin{aligned}
\beta_c^i(s^i, S^i, \bar{L}) \quad = \quad & \Bigg\{ z^i : \sum_{j=M+1}^{n} z_j^i \leq \bar{L} \\
& 0 \leq z_j^i, \quad j = 1,\ldots,M_1, \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M_1+1,\ldots,M, \\
& s_j^i \leq -z_j^i \leq 0, \quad j = (i-M)M+1,\ldots,(i-M)M+M, \\
& z_0^i \geq 0, \\
& z_j^i = 0, \quad \text{for all } \; j \notin \mathcal{A}^i\Bigg\}
\end{aligned}$$

in addition to the budget set $\gamma_c^i(s^i, S^i, q, \tau^i, C^i, \bar{L})$ defined earlier. The first step proves the lower hemicontinuity of the correspondence α_c^i, the next the lower hemicontinuity of the correspondence β_c^i, and finally the lower hemicontinuity of γ_c^i. In the rest of the proof we suppress the index i.

1. *Lower hemicontinuity of* α_c

   We shall first show that $\alpha_c : R_-^M \times R_+^M \to R_+^{n+1}$ is a lower hemicontinuous correspondence for all (s, S) such that $s_j \leq 0 \leq S_j$. To prove lower hemicontinuity consider $z^\circ \in \alpha_c(s^\circ, S^\circ)$ satisfying the conditions mentioned.

Define

$$J_1 = \{j : z_j^\circ = 0\}, \qquad J_2 = \{j : s_j^\circ = -z_j^\circ < 0\},$$
$$J_3 = \{j : 0 < z_j^\circ = S_j^\circ\}, \qquad J_4 = \{j : j \notin J_1 \cup J_2 \cup J_3\}.$$

Consider a sequence $(s^r, S^r) \to (s^\circ, S^\circ)$ where $s_j \leq 0 \leq S_j$. We now wish to find a sequence $\{z^r\}$, $z^r \in \alpha_c(s^r, S^r)$ such that $z^r \to z^\circ$ as $(s^r, S^r) \to (s^\circ, S^\circ)$, the existence of such a sequence guaranteeing the lower hemicontinuity of α_c.

Define

$$\begin{aligned} z_j^r &= z_j^\circ, \quad j \in J_4 \\ &= 0, \quad j \in J_1 \\ &= -s_j^r, \quad j \in J_2 \\ &= S_j^r, \quad j \in J_3 \end{aligned}$$

By construction, $z^r \in \alpha_c(s^r, S^r)$ for all r sufficiently large. Moreover $z^r \to z^\circ$ for all $j \in J_1 \cup J_2 \cup J_3 \cup J_4$. This establishes the lower hemicontinuity of α_c.

2. *Lower hemicontinuity of β_c*

Let a be a vector of zeros and ones such that $az = \sum_{j=M+1}^n z$. Now consider the correspondence $\beta_c : R_-^M \times R_+^M \times R_+ \to R_+^{n+1}$. To prove lower hemicontinuity, let $z^\circ \in \beta_c(s^\circ, S^\circ, \bar{L}^\circ)$ and $(s^r, S^r, \bar{L}^r) \to (s^\circ, S^\circ, \bar{L}^\circ)$. We now wish to find a sequence $\{z^r\}$, $z^r \in \beta_c(s^r, S^r, \bar{L}^r)$ such that $z^r \to z^\circ$ as $(s^r, S^r, \bar{L}^r) \to (s^\circ, S^\circ, \bar{L}^\circ)$.

Since $z^\circ \in \alpha_c(s^\circ, S^\circ)$, there is a subsequence $z^{r_k} \in \alpha_c(s^r, S^r)$ such that $z^{r_k} \to z^\circ$ (as was shown in the first part of the proof). $\bar{L}^\circ > 0$ implies that there also exits $\hat{z} \in \beta_c(s^\circ, S^\circ, \bar{L}^\circ)$ such that $a\hat{z} < \bar{L}^\circ$. Since $\hat{z} \in \alpha_c(s^\circ, S^\circ)$, there is also a sequence $\hat{z}^r \in \alpha_c(s^r, S^r)$ such that $\hat{z}^r \to \hat{z}$.

Define $z^r = \lambda^r z^{r_k} + (1 - \lambda^r)\hat{z}^r$ where λ^r is maximal for $\lambda \in [0, 1]$ such that $az^r \leq \bar{L}^r$. By construction $x^r \in \beta_c(s^r, S^r, \bar{L}^r)$ for all r sufficiently large. Since λ is bounded, it has a limit point $\bar{\lambda}$, so let $\bar{\lambda} < 1$ if possible. Hence the sequence $\{z^r\}$ has a limit point $\bar{z} = \bar{\lambda}z^\circ + (1-\bar{\lambda})\hat{z}$. Since $\bar{\lambda} < 1$, this implies that $\bar{\lambda}az^\circ + (1 - \bar{\lambda})a\hat{z} = \bar{L}^\circ$ (or else $\lambda < 1$ is not maximal). But since $a\hat{z} < \bar{L}^\circ$ this yields that $az^\circ > \bar{L}^\circ$, contradicting $z^\circ \in \beta_c(s^\circ, S^\circ, \bar{L}^\circ)$. Hence $\lambda^r \to 1$ and $z^r \to z^\circ$, which establishes that β_c is lower hemicontinuous.

3. *Lower hemicontinuity of γ_c*

Finally the lower hemicontinuity of the correspondence $\gamma_c : R_-^M \times R_+^M \times R_+^{n+1} \times R_+ \times R_+ \to R_+^{n+1}$ needs to be proved. To do this we need, for any $z^\circ \in \alpha_c(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$ to find a

sequence $\{z^r\}$, $z^r \in \gamma_c(s^r, S^r, q^r, \tau^r, C^r, \bar{L}^r)$, such that $z^r \to z^\circ$ as $(s^r, S^r, q^r, \tau^r, C^r, \bar{L}^r) \to (s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$. This is done in the same manner as in part 2 of the proof under the assumption that $C > 0$ (there then exits $\hat{z} \in \gamma_c(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$ such that $\tau^\circ q^\circ \hat{z} < C^\circ$). This completes the proof of lemma 1.

□

Lemma 2 *The supply and demand functions $z^i(s^i, S^i, q, \tau^i, C^i, \bar{L})$ for the consumers ($i = M+1,\ldots,M+N$) are continuous in prices, initial resources and quantity constraints.*

Proof: In the following we continue to suppress the index i. The claim would be established if, given any sequence $(s^r, S^r, q^r, \tau^r, C^r, \bar{L}^r) \to (s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$, $z^r = z(s^r, S^r, q^r, \tau^r, C^r, \bar{L}^r) \Rightarrow z^r \to z^\circ$ where $z^\circ = z(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$. Define

$$\check{q}_l \quad = \quad \min_j \tau_j^\circ q_j^\circ$$

We introduce ε such that $\varepsilon > 0$ and $\check{q}_l - \varepsilon > 0$. Define

$$q_* \; = \; (q_{j*}), \qquad \text{where } q_{j*} = \check{q}_l - \varepsilon \text{ for all } j.$$

Then $z^r \in \gamma_c(s^r, S^r, q_*, \tau^r, C^\circ + \varepsilon, \bar{L}^r)$ for all r large enough; this follows since

$$q_* z^r < \tau^r q^r z^r \leq C^\circ + \varepsilon$$

for all r large. Since $\gamma_c(s^r, S^r, q_*, \tau^r, C^\circ + \varepsilon, \bar{L}^r)$ is compact[10], z^r must have a convergent subsequence $z^{r_k} \to z^*$ as $k \to \infty$. Assume $z^* \neq z(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$. This implies that there exists a $\tilde{z}$ which is such that $u(\tilde{z}) > u(z^*)$. Since $\tau^r p^r z^r \leq C^r$, $\tau^\circ q^\circ z^* \leq C^\circ$ so that $z^* \in \gamma_c(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$. Since the budget set $\gamma_c(s, S, q, \tau, C, \bar{L})$ is lower hemicontinuous (see lemma 1) there is $\tilde{z}^r \in \gamma_c(s^r, S^r, q^r, \tau^r, C^r, \bar{L}^r)$ such that $\tilde{z}^r \to \tilde{z}$. Continuity of the utility function gives us that $u(z^r) \geq u(\tilde{z}^r) \Rightarrow u(z^*) \geq u(\tilde{z})$[11]. This is a contradiction. So no such $\tilde{z}$ can exist, and consequently $z^* = z(s^\circ, S^\circ, q^\circ, \tau^\circ, C^\circ, \bar{L}^\circ)$. Since this is true for any arbitrary limit point, $z^r \to z^\circ$.

□

[10] $\tau q > 0 \Rightarrow \gamma_c(s, S, q, \tau, C, \bar{L})$ is compact; suppose, to the contrary, that it is not so: since $\gamma_c(s, S, q, \tau, C, \bar{L})$ is closed, there must be a sequence $z^n \in \gamma_c(s, S, q, \tau, C, \bar{L})$ and $z_j^n \to +\infty$. (This is the only possibility, as $\gamma_c(s, S, q, \tau, C, \bar{L})$ is bounded below.) But this means that $\sum q_j z_j \leq C$ must be violated for n large enough. So no such sequence can exist.

[11] Continuity of the preference relationship $\succeq$ is defined as follows. Let $\{z^r\}$ and $\{\bar{z}^r\}$ be two arbitrary sequences such that $z^r \to z$ and $\bar{z}^r \to \bar{z}$. Then $z^r \succeq \bar{z}^r$ implies $z \succeq \bar{z}$.

A.2 Continuity of the firms' Drèze demand and supply functions

We now look at two lemmas concerning the firms which are similar to those for the consumers.

Lemma 3 *The firm's budget correspondence γ_f^i is lower hemicontinuous at every point $(s^i, S^i, q, t_{2i}, C^i, Y^i)$ such that $q_j > 0$ and $0 \leq t_{2i} < 0$, $z_j^i \leq 0 \leq S_j^i$ for all j, $C^i > 0$, and $Y^i \geq 0$.*

Proof: The proof follows mainly the proof of lemma 1 in the preceeding section. The main difference is that there now is no constraint on hours worked but instead a constraint on the firms' output. In the same manner as earlier we define

$$\begin{aligned}
\alpha_f^i(S^i) \quad = \quad & \{z^i : 0 \leq z_j^i, \quad j = 1,\ldots,M_1, \quad i \neq j \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M_1+1,\ldots,M, \quad i \neq j \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M+i, 2M+i, 3M+i, \ldots, MN+i, \\
& z_0^i \geq 0, \\
& z_j^i = 0, \quad \text{for all} \;\; j \notin \mathcal{A}^i\}
\end{aligned}$$

and

$$\begin{aligned}
\beta_f^i(s^i, S^i, Y^i) \quad = \quad & \left\{z^i : 0 \leq f^i(z^i) - Y^i \leq -s_i^i, \right. \\
& 0 \leq z_j^i, \quad j = 1,\ldots,M_1, \quad i \neq j \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M_1+1,\ldots,M, \quad i \neq j \\
& 0 \leq z_j^i \leq S_j^i, \quad j = M+i, 2M+i, 3M+i, \ldots, MN+i, \\
& z_0^i \geq 0, \\
& \left. z_j^i = 0, \quad \text{for all} \;\; j \notin \mathcal{A}^i\right\}
\end{aligned}$$

in addition to the budget set $\gamma_f^i(s^i, S^i, q, \tau^i, C^i, Y^i)$ defined earlier. As in the proof of lemma 1 we prove that the correspondences α_f^i, β_f^i, and γ_f^i are lower hemicontinuous. Also here we suppress the index i.

1. *Lower hemicontinuity of α_f*

   Proof of the lower hemicontinuity of the correspondence $\alpha_f : R_+^{M+N} \to R_+^{n+1}$ follows from the proof of the lower hemicontinuity of the correspondence α_c.

2. *Lower hemicontinuity of β_f*

   Now consider $\beta_f : R_- \times R_+^{M+N} \times R_+ \to R_+^{n+1}$. To prove lower hemicontinuity, let $z^\circ \in \beta_f(s^\circ, S^\circ, Y^\circ)$ and $(s^r, S^r, Y^r) \to (s^\circ, S^\circ, Y^\circ)$. We now wish to find a sequence $\{z^r\}$, $z^r \in$

$\beta_f(s^r, S^r, Y^r)$ such that $z^r \to z^\circ$ as $(s^r, S^r, Y^r) \to (s^\circ, S^\circ, Y^\circ)$. Since $z^\circ \in \alpha_f(S^\circ)$, there is a subsequence $z^{r_k} \in \alpha_f(S^r)$ such that $z^{r_k} \to z^\circ$.

We first consider the situation where the sales constraint is an equality, $f(z) - Y = -s_i$. This occurs when considering a government firm (which also implies that $Y = 0$) or when $s_i = 0$. Define the function $g(z_0) = \{f(z) : z_j = z_j^\circ, j = 1, \ldots, M+N\}$, where z_0 is the numeraire good money. Since $f(z)$ is a monotonic increasing function in z_0, we have that $g(z_0)$ is invertable[12], where the inverse is denoted by g^{-1}. Consequently choose z^r so that

$$\begin{aligned} z_0^r &= g^{-1}(-s_i^r + Y^r) \\ z_j^r &= z_j^\circ \quad \text{for } j \neq 0 \end{aligned}$$

We then have that $z^r \in \beta_f(s^r, S^r, Y^r)$ and that $z^r \to z^\circ$ as $(s^r, S^r, Y^r) \to (s^\circ, S^\circ, Y^\circ)$.

When considering private firms where $s_i^\circ < 0$ we have that there, in addition to z°, exits $\hat{z} \in \beta_f(s^\circ, S^\circ, Y^\circ)$ such that $\hat{z}_j \leq z_j^\circ$ for all j and $\hat{z}_j < z_j^\circ$ for at least one j. Since $\hat{z} \in \alpha_f(S^\circ)$, there is also a sequence $\hat{z}^r \in \alpha_f(S^r)$ such that $\hat{z}^r \to \hat{z}$.

Define $z^r = \lambda^r z^{r_k} + (1-\lambda^r)\hat{z}^r$ where λ^r is maximal for $\lambda \in [0,1]$ such that $f(z^r) - Y^r \leq -s_i^r$. By construction $z^r \in \beta_f(s^r, S^r, Y^r)$. Since λ^r is bounded, it has a limit point $\bar{\lambda}$, so let $\bar{\lambda} < 1$ if possible. Hence the sequence $\{z^r\}$ has a limit point $\bar{z} = \bar{\lambda}z^\circ + (1-\bar{\lambda})\hat{z}$. Since $\bar{\lambda} < 1$, this implies that $f(\bar{\lambda}z^\circ + (1-\bar{\lambda})\hat{z}) - Y^\circ = -s_i^\circ$ (or else $\bar{\lambda} < 1$ is not maximal). $z^\circ > \hat{z}$ implies that $z^\circ > \bar{\lambda}z^\circ + (1-\bar{\lambda})\hat{z}$ when $\lambda < 1$, leading to $f(z^\circ) - Y^\circ > f(\bar{\lambda}z^\circ + (1-\bar{\lambda})\hat{z}) - Y^\circ = -s^\circ$, which contradicts $f(z^\circ) - Y^\circ \leq -s^\circ$. Hence $\lambda^r \to 1$ and $z^r \to z^\circ$, which establishes that β_f is lower hemicontinuous.

3. *Lower hemicontinuity of γ_f*

   The lower hemicontinuity of the correspondence $\gamma_f : R_- \times R_+^{M+N} \times R_+^{n+1} \times R_+^n \times R_+ \times R_+ \to R_+^{n+1}$ is proved by following the proof of the lower hemicontinuity of β_c from lemma 1.

□

Lemma 4 *The supply and demand functions $z^i(s^i, S^i, q, \tau^i, C^i, Y^i)$ for the firms $(i = 1, \ldots, M)$ are continuous in prices, initial resources and quantity constraints.*

Proof: This follows the proof of the continuity of the demand and supply functions of the consumer.

□

[12] The function g is non-singular so that $g(z_0) = 0$ implies that $z_0 = 0$.

A.3 Existence of equilibrium

Using the notation introduced earlier in the appendix, theorem 1 in the main paper can be formulated as follows. Note that $\sum_{j=1}^{M} z_k^j(s^j, S^j, q, t_{2i}, C^j, Y^j) = l_{ij}^d$ where $i = {}_{(k-j)/M+M}$ and $\sum_{i=M+1}^{M+N} z_k^i(s^i, S^i, q, \tau^i, C^i, \bar{L}) = l_{ij}^s$ where $j = {}_{k-(i-M)M}$. Since we are dealing with an open economy, condition 1.2 below only considers the markets for the non-tradeable goods (including labor).

Theorem 1 *Given any (q, τ^i, C^i, Y^i) such that $q_j > 0$ for all j while $\tau^i > 0$, $1 > t_{2i} \geq 0$, $C^i \geq 0$, and $Y^i \geq 0$ for all i, there exist maximum and minimum constraints (s^i, S^i) satisfying*

1.1 $s_i^i \leq 0, \qquad i = {}_{1,\ldots,M}$

$0 \leq S_j^i, \qquad j = {}_{M+i,2M+i,3M+i,\ldots,MN+i} \qquad$ *for all* $i = {}_{1,\ldots,M}$,

$s_j^i \leq 0, \qquad j = {}_{(i-M)M+1,\ldots,(i-M)M+M} \qquad$ *for all* $i = {}_{M+1,\ldots,M+N}$,

$0 \leq S_j^i, \qquad j = {}_{M_1+1\ldots,M} \qquad$ *for all* $i = {}_{1,\ldots,M+N}$, $i \neq j$

1.2 $y^j(s^i, S^i, q, t_{2i}, C^i, Y^i) - \sum_{i=1}^{M} z_j^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$
$- \sum_{i=M+1}^{M+N} z_j^i(s^i, S^i, q, \tau^i, \bar{L}, C^i) = 0, \qquad j = {}_{M_1+1,\ldots,M_2}$,

$y^j(s^i, S^i, q, t_{2i}, C^i, Y^i) - \sum_{i=M+1}^{M+N} z_j^i(s^i, S^i, q, \tau^i, \bar{L}, C^i) = 0, \qquad j = {}_{M_2+1,\ldots,M}$,

$\sum_{i=M+1}^{M+N} z_j^i(s^i, S^i, q, \tau^i, C^i, \bar{L}) - \sum_{i=1}^{M} z_j^i(s^i, S^i, q, t_{2i}, C^i, Y^i) = 0, \quad j = {}_{M+1,\ldots,M+MN}$,
where $z^i(s^i, S^i, q, \tau^i, C^i, \bar{L})$ for $i = {}_{M+1,\ldots,M+N}$ are the Drèze demands and supplies which solve the problem

$$
\begin{aligned}
\max \quad & U^i(z^i) \\
\text{s.t.} \quad & \tau^i q z^i \leq C^i, \\
& \sum_{j=1}^{NM} z_j^i \leq \bar{L}, \\
& 0 \leq z_j^i, \quad j = {}_{1,\ldots,M_1}, \\
& 0 \leq z_j^i \leq S_j^i, \quad j = {}_{M_1+1,\ldots,M}, \\
& s_j^i \leq -z_j^i \leq 0, \quad j = {}_{(i-M)M+1,\ldots,(i-M)M+M}, \\
& z_0^i \geq 0, \\
& z_j^i = 0, \quad \textit{for all } \; j \notin \mathcal{A}^i
\end{aligned}
$$

and $z^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$ and $y^i(s^i, S^i, q, t_{2i}, C^i, Y^i)$ for $i = {}_{1,\ldots,M}$ are the Drèze demands

and supplies which solve the problem

$$\begin{aligned}
\max \quad & \pi^i(z^i) \\
\text{s.t.} \quad & qz^i \le C^i - \sum_j v_{1j} inv_{ji}, \\
& s^i_i \le -y^i + Y^i \le 0, \\
& 0 \le z^i_j, \quad j = 1,...,M_1, \quad j \ne i, \\
& 0 \le z^i_j \le S^i_j, \quad j = M_1{+}1,...,M_2, \quad j \ne i, \\
& 0 \le z^i_j \le S^i_j, \quad j = M{+}i, 2M{+}i, 3M{+}i, ..., MN{+}i, \\
& z^i_0 \ge 0, \\
& z^i_j = 0, \quad \textit{for all } \; j \notin \mathcal{A}^i
\end{aligned}$$

1.3 *1.* $-y^i(s^i, S^i, q, t_{2i}, C^i, Y^i) = s^i_i$ *for some* i *implies that* $z^h_i(s^h, S^h, q, t_{2i}, C^h, Y^h) < S^h_i$ *for all* $h = 1,...,M$ *and* $z^h_i(s^h, S^h, q, \tau^h, C^h, \bar{L}) < S^h_i$ *for all* $h = M{+}1,...,M{+}N$*;*

2. $z^i_j(s^i, S^i, q, t_{2i}, C^i, Y^i) = S^i_j$ *for some* $i = 1,...,M$ *and* $j = M_1{+}1,...,M$ *or* $z^i_j(s^i, S^i, q, \tau^i, C^i, \bar{L}) = S^i_j$ *for some* $i = M{+}1,...,M{+}N$ *and* $j = M_1{+}1,...,M$ *implies that* $-y^j(s^j, S^j, q, t_{2j}, C^j, Y^j) > s^j_j$*;*

3. $-z^i_j(s^i, S^i, q, \tau^i, C^i, \bar{L}) = s^i_j$ *for some* $i = M{+}1,...,M{+}N$ *implies that* $z^h_j(s^h, S^h, q, t_{2h}, C^h, Y^h) < S^h_j$ *for* $h = j(i{-}M)M$*;*

4. $z^i_j(s^i, S^i, q, t_{2i}, C^i, Y^i) = S^i_j$ *for some* $i = 1,...,M$ *implies that* $-z^h_j(s^h, S^h, q, \tau^h, C^h, \bar{L}) > s^h_j$ *for* $h = (j{-}i)/M{+}M$*.*

Proof: Let f^{i*} be a bound on the output of firm i,

$$f^{i*} \quad = \quad \{y^i : y^i = f^i(z^i), z^i_j = C^i/q_j, {}_{j=1,...,M}, z^i_j = T, {}_{j=M+i,2M+i,...,MN+i}\},$$

for all $i = 1,...,M$. Furthermore let

$$C^{i*} \quad = \quad \begin{cases} C^i - \sum_k v_{1k}\, inv_{ki}, & \text{when } \; i = 1,...,M, \\ C^i + (\max_j w_{ij})\bar{L}, & \text{when } \; j = M{+}1,...,M{+}N, \end{cases}$$

$$V_j \quad = \quad \begin{cases} \max_i (C^{i*}/q_j) + f^{i*} + \varepsilon, & \text{when } \; j = 1,...,M, \\ \max_i (C^{i*}/q_j) + \bar{L} + \varepsilon, & \text{when } \; j = M{+}1,...,M{+}MN, \end{cases}$$

$$\begin{aligned}
I_{J_1} &= [-V_j, 0], \\
I_{J_2} &= [0, V_j], \\
I_1 &= \times_{j=1}^n I_{J_1}, \\
I_2 &= \times_{j=M_1+1}^n I_{J_2},
\end{aligned}$$

where $\varepsilon > 0$ (and $n = {}_{M+MN}$), $s^i \in I_1$, $S^i \in I_2$, and $(s^i, S^i) \in I_1 \times I_2 = I$. Thus,

$$\left\{(s^1, S^1), (s^2, S^2), \ldots, (s^{N+M}, S^{N+M})\right\} \in I^{N+M}.$$

Define

$$\begin{aligned} I^* = [\{(s^i, S^i)\} : \quad & \{(s^i, S^i)\} \in I^{M+N} \text{ and } S^i_j - s^j \geq V_j, \;\; j = {}_{M_1+1,\ldots,M_2}, \; i = {}_{1,\ldots,M}, \\ & \text{and } S^i_j - s^j \geq V_j, \;\; j = {}_{M_1+1,\ldots,M}, \; i = {}_{M+1,\ldots,M+N}, \\ & \text{and } S^i_j - s^h_j \geq V_j, \;\; j = {}_{M+1,\ldots,M+N}, h = {}_{(j-1)/M+M}]. \end{aligned}$$

The set I^* is non-empty, compact and convex. Following Mukherji (1990) for each $\{(s^i, S^i)\} \in I^*$ define $Q(\{(s^i, S^i)\}) = \{(s^{i'}, S^{i'})\}$ by

$$s^{i'}_j = \begin{cases} s^i_j & \text{if } \; j = {}_{1,\ldots,M_1}, \\ \dfrac{s^i_j}{1+ \mid k_j(s^i, S^i) \mid} & \text{if } \; k_j(s^i, S^i) \leq 0 \;\text{ and }\; j = {}_{M_1+1,\ldots,M+MN}, \\ \dfrac{s^i_j - V_j \cdot \mid k_j(s^i, S^i) \mid}{1+ \mid k_j(s^i, S^i) \mid} & \text{otherwise ,} \end{cases}$$

$$S^{i'}_j = \begin{cases} \dfrac{S^i_j + V_j \cdot \mid k_j(s^i, S^i) \mid}{1+ \mid k_j(s^i, S^i) \mid} & \text{if } \; k_j(s^i, S^i) \leq 0, \\ \dfrac{S^i_j}{1+ \mid k_j(s^i, S^i) \mid} & \text{otherwise ,} \end{cases}$$

where

$$k_j(s^i, S^i) = \begin{cases} y_j(\cdot) - \displaystyle\sum_{i=1}^{M} z^i_j(\cdot) - \sum_{i=M+1}^{M+N} z^i_j(\cdot), & j = {}_{M_1+1,\ldots,M}. \\ \displaystyle\sum_{i=1}^{M} z^i_j(\cdot) - \sum_{i=M+1}^{M+N} z^i_j(\cdot), & j = {}_{M+1,\ldots,M+MN} \end{cases}$$

Notice that $\{(s^{i'}, S^{i'})\} \in I^N$, and

$$\begin{aligned} S^{i'}_j - s^{k'}_j &= \frac{S^i_j - s^k_j + V_j \cdot \mid k_j(s^i, S^i) \mid}{1+ \mid k_j(s^i, S^i) \mid} \\ &\geq V_j \;\text{ if }\; \{(s^i, S^i)\} \in I^* \end{aligned}$$

for $j > M_1$. We have thus constructed Q so that $Q : I^* \to I^*$. The continuity of $k_j(\cdot)$ established by virtue of lemma 2 and 4 implies that Q is a continuous function from I^* to itself. Hence by virtue of Brouwer's Fixed-Point Theorem, there is $\{(s^{i*}, S^{i*})\} \in I^*$ such that

$$Q[\{(s^{i*}, S^{i*})\}] = \{(s^{i*}, S^{i*})\}.$$

We have thereby found that there exist maximum and minimum constraints for any given (q, τ^i, C^i, Y^i). We now show that conditions 1.1 to 1.3 hold at $\{(s^{i*}, S^{i*})\}$. Suppose that $k_j(s^{i*}, S^{i*}) < 0$ for some $j > M_1$. Then we get that

$$s_j^{i*} = \frac{s_j^{i*}}{1+ \mid k_j(\cdot) \mid} \quad \Rightarrow \quad s_j^{i*} = 0$$

for all i. Using the definition of $k_j(\cdot)$ this implies that $k_j(\cdot) \geq 0$, which is a contradiction. Hence $k_j(s^{i*}, S^{i*}) \geq 0$ for all $j > M_1$. Suppose that the strict inequality holds for some j. Then

$$S_j^{i*} = \frac{S_j^{i*}}{1+ \mid k_j(\cdot) \mid} \quad \Rightarrow \quad S_j^{i*} = 0 \ \Rightarrow \ k_j(\cdot) \leq 0 :$$

a contradiction once again. Hence $k_j(s^{i*}, S^{i*}) = 0$ for all $j \neq 0$, implying that $\sum_i z_0(s^{i*}, S^{i*}) = \sum_i C^i - T - ts$. Thus, conditions 1.1 and 1.2 in theorem 1 hold at $\{(s^{i*}, S^{i*})\}$.

The proof is completed by showing that condition 1.3 also holds at $\{(s^{i*}, S^{i*})\}$. First, suppose that for some $j = {}_{M_1+1,\ldots,M}$ there exits i such that

$$z_j^i(s^{i*}, S^{i*}, q, \tau^i, C^i) = S_j^{i*}$$

and

$$-y^j(s^{j*}, S^{j*}, q, \tau^j, C^j) = s_j^{j*}.$$

Then

$$z_j^i(\cdot) \quad = \quad S_j^{i*} \leq C^i/q_j + f^{j*} + s_j^{j*}$$

since $-y^j(\cdot) = s_j^{j*}$ implies that $f^{j*} + s_j^{j*} > 0$. Therefore

$$C^i/q_j + f^{j*} \geq S_j^{i*} - s_j^{j*} \geq V_j$$

or

$$0 \geq V_j - C^i/q_j - f^{j*} \geq \varepsilon > 0,$$

which is a contradiction. Part 1 of condition 1.3 holds trivially for any s_i^i when $i = 1, \ldots, M_1$ since no demanders are ever rationed in these tradeable goods. We therefore have that parts 1 and 2 of condition 1.3 must hold. Finally, suppose that for some $j = {}_{M+1,\ldots,M+MN}$ there exits $i \in {}_{[1,\ldots,M]}$, $k \in {}_{[M+1,\ldots,M+N]}$ such that

$$z_j^i(s^{i*}, S^{i*}, q, \tau^i, C^i) = S_j^{i*}$$

and

$$-z_j^k(s^{k*}, S^{k*}, q, \tau^i, C^k) = s_j^{k*}.$$

Then

$$z_j^i(\cdot) \quad = \quad S_j^{i*} \leq C^i/q_j + \bar{L} + s_j^{k*}, \text{ since } \bar{L} + s_j^{k*} > 0.$$

Therefore

$$C^i/q_j + \bar{L} \geq S_j^{i*} - s_j^{k*} \geq V_j$$

or

$$0 \geq V_j - C^i/q_j - \bar{L} \geq \varepsilon > 0,$$

which again is a contradiction. Hence condition 1.3 in theorem 1 must hold at $\{(s^{i*}, S^{i*})\}$. This completes the proof of the proposition.

□