



Via Po, 53 – 10124 Torino (Italy)  
Tel. (+39) 011 6704043 - Fax (+39) 011 6703895  
URL: <http://www.de.unito.it>

## WORKING PAPER SERIES

**Mortality, fertility and elderly care in a gendered growth model**

Leif Andreassen

Dipartimento di Economia "S. Cagnetti de Martiis"

Working paper No. 10/2010



Università di Torino

# Mortality, fertility and elderly care in a gendered growth model

Leif Andreassen<sup>1 2</sup>

<sup>1</sup>Department of Economics "S. Cogneetti de Martiis", University of Turin, Via Po, 53 10124 Torino

<sup>2</sup>Statistics Norway, P.O.B. 8131 Dep., N-0033 Oslo, Norway

## Abstract

The paper studies the interaction of publicly provided care for the elderly on demographic developments. A two-sex OLG model is used to examine how exogenous changes in mortality, the cost of children and the bargaining power of women influence fertility, public and private care for the elderly, and the length of education taken by women and men. The paper focuses especially on the interaction between declining mortality and the expansion of care for the elderly. In the model declining mortality can affect fertility differently according to how developed the economy is. At an early development stage, when public care is little developed, the effect of decreasing mortality on fertility can be positive, while at a later stage with higher levels of public care, the effect can be negative. The model is consistent with observed developments over the last century including fluctuations and decline in fertility, increases in the average age of giving birth, increasing levels of education with lessening differences in the education levels of women and men, increasing incomes, and increased public care for the elderly. In a small open economy where individuals live for five periods with uncertain lifetimes, the choices made by males and females are the result of a combination of utility maximization and negotiation. First, bargaining positions are formed through utility maximization given individual budget constraints, then the Nash bargaining solution determines the number of children and voting determines the level of public care for the elderly, and finally couples maximize a joint household welfare function to determine education, private care for the elderly and consumption. The only exogenous differences between women and men concern mortality, bargaining power and required time devoted to raising young children; otherwise women and men have identical utility functions and opportunities. Functional forms are chosen so that the model has a recursive nature with simple closed form solutions.

Keywords: mortality, fertility, old age care, olg      JEL: D1 D9 J1

# 1 Introduction

Present concerns about whether aging societies can adequately provide for their elderly have their basis in historical changes in mortality and fertility along with expansions in public pension systems and publicly provided care for the elderly. While the effects of establishing public pension systems have been much studied, there has been less research on the role government provided care has played. Increases in public care have for example allowed individuals to provide less private care and to be less dependent on their children in old age. Changes in educational levels, productivity, and income are important underlying factors in understanding the expansion in the public sector and in determining the seriousness of the challenges presented by an aging population. In the paper a two-sex overlapping-generations model is presented that examines how exogenous changes in mortality, the cost of children and the bargaining power of women influence fertility, public and private care for the elderly, and the length of education taken by women and men. The paper focuses especially on the interaction between declining mortality and the expansion of care for the elderly. For example declining mortality can affect fertility differently according to how developed the economy is. At an early development stage, when public care is little developed, the effect of decreasing mortality on fertility can be positive, while at a later stage with higher levels of public care, the effect can be negative. The model is consistent with observed developments over the last century including fluctuations and decline in fertility, increases in the average age of giving birth, increasing levels of education with lessening differences in the education levels of women and men, increasing incomes, and increased public care for the elderly. The framework employed is an overlapping-generations model of a small open economy where individuals live for five periods with uncertain lifetimes. The choices made by males and females in each generation are the result of a combination of utility maximization and negotiation. First, bargaining positions are formed through utility maximization given individual budget constraints, then the Nash bargaining solution determines the number of children and voting determines the level of public care for the elderly, and finally couples maximize a joint household welfare function to determine education, private care for the elderly and consumption. The only exogenous differences between women and men concern mortality, bargaining power and required time devoted to raising young children; otherwise women and men have identical utility functions and opportunities. Functional forms are chosen so that the model has a recursive nature with simple closed form solutions.

During the last century fertility has fallen, risen, and fallen again. At present it is at a level close to, but below, the reproduction level of 2.1 children per women. The number of live births per women in Norway fell from 4.5 for the cohort born 1850 to 1.96 for the cohort born 1905. After this it rose again to 2.58 for the cohort born in 1934. Since then the number of live births

have fallen to 2.09 for the cohort born in 1950 and is expected to further fall to the present total fertility rate of around 1.86.

At the same time as fertility was fluctuating, the timing of births has been changing and the educational attainment level of the population has been increasing, with a narrowing of the gap between the educational level of men and women. Average age of birth was 29.39 years of age for the Norwegian cohort born in 1900, sinking to 25.9 years of age for the 1945 cohort and then rising to 27.88 years for the 1978 cohort. The average age at first birth shows a similar development pattern, falling from 23.4 years of age for the Norwegian cohort born in 1935 to 22.8 years of age for the cohort born in 1950, before rising again to 24.8 years of age for the cohort born in 1958.

From 1962 to 1992 the proportion of 16-year-olds under education has increased from 53.8 percent to 93.5 percent and the proportion of 20-year-olds under education has increased from 16.6 percent to 43.8 percent. Women are now in the majority at universities as well as at colleges.

The model presented in the following will attempt to explain these developments as arising from: i) Decreasing mortality; ii) Increasing costs (decreasing economic benefits) of older children as Norway became increasingly urbanized; iii) Decreasing time costs for women connected with caring for young children, approaching the time costs of men, due to labor saving devices in the home; iv) Increasing bargaining strength of women relative to men with regard to fertility decisions.

During the last 15 years a large literature has evolved discussing different aspects of fertility choice within a dynamic framework. Excellent surveys of the literature can be found in for example Holz, Klerman and Willis (1997) or Arroyo and Zhang (1997). The paper is closely related to a recent paper by Blackburn and Cipriani (2002), where economic and demographic outcomes are jointly determined in a dynamic general equilibrium model of longevity, fertility and growth. While longevity is assumed to be exogenous in the following, this paper extends their analysis by analyzing the interaction between exogenously given mortality, and endogenous fertility and growth in publicly provided old age care. As in their paper, parents are non-altruistic, deriving utility only from the production of children and not the children's welfare, but the present paper also includes utility derived from the care received children. Mortality is modelled as in Blackburn and Cipriani (2002), including only old age mortality with no consideration of infant mortality (which also is an important determinant of fertility).

Another related paper discussing morality and fertility is Yakita (2001), which finds that an increase in life expectancy lowers the fertility rate and raises life-cycle savings, and that pay-as-you-go social security does not reverse the effect on fertility. This is the result of worker wanting more consumption as old, at the cost of reducing other consumption, including consumption of children. This paper shows that by including old age care, this is no certain conclusion, depending on the extent to which there is a publicly provided pay-as-you-go old age care system.

The paper explicitly considers the timing of births as well as the number of children born because it is thought that changes in the cost of raising children at different ages is an important determinant of fertility. As such it complements Iyigun (2000) who finds that increases in the human capital stock raise the opportunity cost of having children while young and induce individuals to delay childbearing. In the following, the average age at birth is also influenced by the relative negotiating strength of women regarding fertility and how strong seniority effects are. Increased negotiation strength normally leads to an increase in the average age at birth, but if seniority counts for little this need not be the case.

A fairly novel aspect of the model is that it is a two-sex model, reflecting the fact that fertility is the result of the actions of two people, each with their own interests. The paper employs a Nash bargaining solution to ensure consistency between the desires of females and males within the household. There is no sorting, as in for example Fernández (2002) which develops a model of skill development and marital sorting. A sophisticated treatment of marriage and divorce within an overlapping generations model (but without fertility) is provided in Aiyagari, Greenwood and Guner (2000).

## 2 The general economic environment

The framework employed in the following is an overlapping-generations model of a small open economy where individuals live for five periods with uncertain lifetimes. All individuals survive through the first four periods, but have a probability of dying before the fifth retirement period. Time is discrete and indexed by  $t$ , while generation  $i$  is defined as those born at time  $t = i$ . The age of an individual of generation  $i$  is thereby given by  $t - i$  with ages numbered from 0 to 4.

Age 0 consists of childhood, education is taken at age 1, children are born at ages 1 and 2, work extends over the ages from 1 to 3, and retirement is at age 4. Each child is cared for over two periods, and parents are cared for when they reach age 4.

When retiring at age 4 parents leave bequests to their children. Bequests have a voluntary component and an involuntary component connected with dying before one reaches age 4. It should be noted that even though individuals are economically active at age 1 (taking an education and having children) they are still being cared for by their parents since care of children spans two periods. This is a result of the long time interval each age spans.

In the model it is assumed that the time cost of young children differs between the sexes, especially due to women's extra time costs connected with carrying a child and breast feeding it. An increase in the net time cost (taking into account work done by children in the household) will decrease the number of children. The distribution in the cost of having children over the life cycle

determines the spacing of children. An increase in the cost at low ages relative to higher ages, will lead to a postponement of births.

Changes in mortality will make the utility of being cared for by one's children when one is old change over time. The longer one lives in old age, the greater is the utility of being cared for and the more children one wishes to have. Since the mortality of men is generally greater than that of women, this effect will in isolation lead to men desiring less children than women.

Individuals can be cared for by a combination of children and the government. As incomes increase individuals wish to substitute care provided by children with publicly provided care.

Production is the product of a constant returns function in real capital and skills subject to exogenous technological progress. Skills are assumed to be the product of an age specific experience parameter and the level of human capital. The accumulation of human capital is assumed to be a function of earlier generations' level of human capital and the amount of education undertaken. A small open economy is assumed, so that the real interest rate is given exogenously by international capital markets.

## 2.1 Old age care

Changes over time in individuals' utility of old age is explicitly modeled. Let  $n_{1ik}$  and  $n_{2ik}$  be the desired number of child births at respectively age 1 and age 2 for individuals of type (sex)  $k$  in generation  $i$ . Total time used on old age care given by each child is denoted  $E_{ik}^*$  while old age care supplied by the government to individuals in generation  $i$  is denoted  $G_i^*$ . The subutility of old age care for individuals of type  $k$  in generation  $i$  is assumed to be given by

$$u_{E ik} = (n_{1ik} \cdot E_{ik}^*)^{\alpha_{E i}/2} \cdot (n_{2ik} \cdot E_{ik}^*)^{\alpha_{E i}/2} \cdot (1 + G_i^*)^{\alpha_G} \quad (1)$$

where  $\alpha_{E i}$  and  $\alpha_G$  are assumed to be parameters. The parameter  $\alpha_{E i}$  is assumed to vary over time according to

$$\alpha_{E i} = \begin{cases} \xi_1 + \xi_2 \cdot (1 + \bar{G}_{i-4})^{-1} & \text{if } 0 \leq \bar{G}_{i-4} < \hat{G} \\ \xi_1 & \text{if } \bar{G}_{i-4} \geq \hat{G} \end{cases}, \quad (2)$$

with  $\hat{G}$  being a high level of government provided care. The utility of old age care received from ones' children decreases with increasing government provided care as long as the level of government care  $\hat{G}$  is less than  $\bar{G}_{i-4}$  ( $\bar{G}_{i-4}$  is the level of care observed at age 0 by generation  $i$ ).

The above subutility function can be seen as being derived from an underlying Cobb-Douglas utility function in physical and emotional care when care is aggregated using Cobb-Douglas type aggregators. Assume that the aggregate amount of total care given by all one's children can be written

$$(n_{1ik} \cdot E_{ik}^*)^{\xi_1^*} \cdot (n_{2ik} \cdot E_{ik}^*)^{\xi_2^*},$$

and that the physical care,  $E_{1ik}^*$ , and the emotional care,  $E_{2ik}^*$ , given by each child are of a joint production nature with each being a proportion,  $\xi_0$  and  $1 - \xi_0$  respectively, of total care,  $E_{ik}^*$ , so that

$$\begin{aligned} E_{1ik}^* &= \xi_0 \cdot \left[ (n_{1ik} \cdot E_{ik}^*)^{\xi_1^*} \cdot (n_{2ik} \cdot E_{ik}^*)^{\xi_1^*} \right], \\ E_{2ik}^* &= (1 - \xi_0) \cdot \left[ (n_{1ik} \cdot E_{ik}^*)^{\xi_1^*} \cdot (n_{2ik} \cdot E_{ik}^*)^{\xi_1^*} \right]. \end{aligned}$$

Taking into account that the government supplements the private provision of physical care by supplying  $G_i^*$ , assume that the subutility function is given by

$$u_{E ik}^* = (E_{2ik}^*)^{\xi_2^*} \cdot \left( (1 + E_{1ik}^*)^{\zeta_1 + \zeta_2 / (1 + G_{i-4})} (1 + G_i^*)^{\zeta_3 + \zeta_4 / (1 + E_{1,i-4,k}^*)} \right)^{\xi_3^*},$$

so that if there is a low level of either privately or publicly provided physical care, the parameter pertaining to the utility of the abundantly provided type care is higher than it otherwise would be. In other words, at low levels of either type of care the utility function is no longer fully separable. Finally assume that  $E_{1ik}^*$  is always fairly large and dividing by the predetermined constant  $\xi_{0i}^*$ , the subutility function  $u_{E ik}^*$  can approximately be written

$$u_{E ik}^* = \frac{1}{\xi_{0i}^*} (E_{2ik}^*)^{\xi_2^*} \cdot \left( (E_{1ik}^*)^{\zeta_1 + \zeta_2 / (1 + G_{i-4})} (1 + G_{ik}^*)^{\zeta_3} \right)^{\xi_3^*}.$$

Inserting for  $E_{1ik}^*$  and  $E_{2ik}^*$  gives the subutility function (1) with

$$\alpha_G = \zeta_3 \xi_3^*, \quad \xi_1 / 2 = \xi_1^* \xi_2^* + \xi_1^* \xi_3^* \zeta_1, \quad \xi_2 / 2 = \xi_1^* \xi_3^* \zeta_2,$$

and

$$\xi_{0i}^* = (1 - \xi_0)^{\xi_2^*} \cdot \xi_0^{(\zeta_1 + \zeta_2 / (1 + G_{i-4})) \xi_3^*}.$$

## 2.2 The individual utility function

Individuals of type  $k$  belonging to generation  $i$  consume  $c_{jik}$  at age  $j$  and desire to accumulate wealth,  $W_{ik}$ , both as a status signal and to be able to leave bequests to their children. The individual expected lifetime utility of these individuals,  $U_{ik}$ , depends on consumption, accumulated wealth and two subutility components connected with having children. It is assumed to have the logarithmic form

$$\begin{aligned} U_{ik}(\pi_{i+4,k}) &= \ln c_{1ik} + \beta \ln c_{2ik} + \beta^2 \ln c_{3ik} + \pi_{i+4,k} \beta^3 \ln c_{4ik} + \alpha_W \ln W_{ik} \\ &\quad + \ln u_{nik} + \pi_{i+4,k} \ln u_{E ik} \end{aligned} \quad (3)$$

where  $\beta$  and  $\alpha_B$  are parameters,  $\pi_{t+4,k}$  is the probability of surviving to the fifth period (age 4),  $u_{nik}$  is the direct utility derived from having children and  $u_{E ik}$  is the utility of being cared for

during retirement as discussed in the previous section. Letting  $\alpha_0$  be a parameter, the subutility function describing the direct utility of children is given by:

$$u_{nik} = (n_{1ik})^{\alpha_0/2} \cdot (n_{2ik})^{\alpha_0/2}, \quad (4)$$

where  $\alpha_0/2$  is the elasticity of having children at either age 1 or age 2.

The preference structure above is completely symmetrical in the parameters concerning children born at an early or a late stage in life so there is no inherent preference for either  $n_{1i}$  or  $n_{2i}$ . The logarithmic structure above assumes that  $n_{1i}$  and  $n_{2i}$  are not perfect substitutes, with agents instead having preferences for spacing of their children. Notice that all the parameters in the utility function are gender neutral, only the longevity variable differs between men and women.

The Cobb-Douglas structure of the utility function implies a direct connection between budget shares and the parameters of the utility function adjusted for mortality. To simplify notation later, from the sum of coefficients

$$\Lambda_{ik}^* = 1 + \beta + \beta^2 + \pi_{i+4,k}\beta^3 + \pi_{i+4,k}\alpha_{Ei} + \alpha_W + \alpha_0 + \pi_{i+4,k}\alpha_{Ei},$$

the following share parameters are defined

$$\begin{aligned} \Lambda_{Nik} &= \frac{1}{\Lambda_{ik}^*} \alpha_0, & \Lambda_{Wik} &= \frac{1}{\Lambda_{ik}^*} \alpha_W, \\ \Lambda_{Eik} &= \frac{1}{\Lambda_{ik}^*} (\pi_{i+4,k} \alpha_{Ei}), \end{aligned}$$

where  $\Lambda_{Nik}$  is the share parameter connected with having children,  $\Lambda_{Eik}$  with receiving care from one's children when old and  $\Lambda_{Wik}$  with accumulating wealth that is left as a bequest to one's children.

## 2.3 Children

The choices made by males and females in each generation are the result of utility maximization and negotiation. Negotiation between the sexes is assumed to happen according to the following sequence:

1. Each person's initial negotiating position is found by maximizing lifetime utility with respect to children subject to their individual budget constraint. This will be their first best notional demand for children.
2. The couple negotiate, agreeing on the number of children they wish to have, the effective demand for children. In addition the level of government provided care is determined.
3. Given the agreed effective demand for children, they maximize a joint household welfare function to find their conditional optimal levels of education, care for the elderly and wealth accumulation (which is left as a bequest to one's children).



Before entering into negotiations about the number of children they are going to have, the representative female and male of generation  $i$  each maximize utility given their individual budget constraints. This leads to male and female notional demands for children born early at age 1, denoted  $\tilde{n}_{1im}$  and  $\tilde{n}_{1if}$ , and notional demands for children born late at age 2, denoted  $\tilde{n}_{2im}$  and  $\tilde{n}_{2if}$ . After determining their notional demands for children, the couple negotiates common effective demands  $\bar{n}_{1i}$  and  $\bar{n}_{2i}$ . The total number of children actually born by generation  $i$  will thereby be  $\bar{n}_{1i} + \bar{n}_{2i}$ . Since these children are born in two different time periods they will belong to two different cohorts. The size of cohort  $i$ , denoted  $N_i$ , will thereby be the sum of children born early by the previous generation  $i - 1$  and children born late by generation  $i - 2$ ,  $N_i = \bar{n}_{1,i-1} + \bar{n}_{2,i-2}$ .

Individual  $k$ 's notional demand for wealth accumulation,  $\tilde{W}_{ik}$ , is determined in the first optimization stage and the couples common effective accumulation,  $\bar{W}_i$ , is determined after negotiation about children.

## 2.4 Individual and household budget constraints

At each working age (ages 1 to 3) individuals have at their disposal  $T$  hours which can be used for work, education and taking care of children. Education is taken at age 1, with the education taken by individual of type  $k$  of generation  $i$  being denoted  $e_{ik}$  (where the subscript  $k$  denotes male or female).

Children are cared for over two periods with the time used per child varying between the sexes for care of young children (of age 0), but not for care of older children (children of age 1). The time used caring for each younger child by generation  $i$  is denoted  $\varphi_{1if}$  and  $\varphi_{1im}$ , while the time used per older child is denoted  $\varphi_{2i}$ . Time usage is constant within a generation, but can change between generations due to changes in housekeeping technologies (the introduction of the washing machine for example). In addition, it is possible to interpret the time parameters as net time costs, being the difference between the time used on children and the time children themselves are productively employed in the household. In the following, it will be supposed that the cost of older children,  $\varphi_{2i}$ , has been increasing both because urbanization has lead them to contribute less to the household and because of increases in the cost of their upkeep. On the other hand it is supposed that the cost of young children faced by women,  $\varphi_{1if}$ , has been falling and converging towards that of men due to technological advances.

Care provided to parents by individuals of type  $k$  in generation  $i$  is denoted  $E_{ik}$  if privately supplied and  $G_{ik}$  if supplied indirectly by the government through taxes (earlier we defined  $E_{ik}^*$  and  $G_{ik}^*$  as the amount of care received). Care is only given if parents do not die before reaching age 4. This means individuals will supply  $\pi_{i+2}^* \eta_{2i} \cdot (E_{ik} + G_{ik})$  of care at age 2 and  $\pi_{i+3}^* \eta_{1i} \cdot (E_{ik} + G_{ik})$

at age 3. The costs at time  $t$  of taking care of one's parents is denoted  $p_{Et}$  and  $p_{Gt}$  respectively.

Notice that care of children and parents are not treated symmetrically. The cost of raising children is connected to time use and thereby explicitly to the endogenous wage, while it is assumed that agents treat the cost of taking care of parents as an exogenous price.

The total bequest received by individuals of generation  $i$  with young parents (born at time  $i - 1$ ) is denoted  $B_{i,i-1,k}$  and the the total bequest received by those with older parents (born at time  $i - 2$ ) is denoted  $\hat{B}_{i,i-2,k}$ .

Letting  $r_t$  be the interest rate in period  $t$  and denoting  $c_{jik}$  as the consumption,  $w_{jik}$  as the wages, and  $s_{jik}$  as the savings of individuals of type  $k$  of generation  $i$  at age  $j$ , with  $k$  indexing sex ( $k = f, m$ ), the individual constraints faced by males and females of generation  $i$  can be written

$$\begin{aligned}
c_{1ik} &= w_{1ik} (T - \Gamma_i \cdot e_{ik} - \varphi_{1ik} n_{1ik}) - s_{1ik} \\
c_{2ik} &= w_{2ik} (T - \varphi_{1ik} n_{2ik} - \varphi_{2i} n_{1ik}) - p_{E,i+2,k} \cdot \pi_{i+2}^* \eta_{2i} E_{ik} - p_{G,i+2} \cdot \pi_{i+2}^* \eta_{2i} G_{ik} \\
&\quad + (1 + r_{i+2}) s_{1ik} - s_{2ik} + \hat{B}_{i,i-2,k} \\
c_{3ik} &= w_{3ik} (T - \varphi_{2i} n_{2ik}) - p_{E,i+3,k} \cdot \pi_{i+3}^* \eta_{1i} E_{ik} - p_{G,i+3} \cdot \pi_{i+3}^* \eta_{1i} G_{ik} \\
&\quad + (1 + r_{i+3}) s_{2ik} - s_{3ik} + \hat{B}_{i,i-1,k} \\
c_{4ik} &= (1 + r_{i+4}) s_{3ik} - W_{ik}
\end{aligned} \tag{5}$$

where  $\Gamma_i$  is a variable describing the difficulty (time cost) of acquiring a specified amount of education given by

$$\Gamma_i = \exp(\Gamma_{i0}) \cdot (h_{i-1})^{-\Gamma_{i1}}$$

with  $\Gamma_{i0} > 0$  and  $0 < \Gamma_{i1} < (1 - \theta_1) / \theta_2$ , so that a high  $h_{i-1}$  makes it easier to learn.

It should be noted that in the above formulation the cost of children increases proportionally with income (wages). It is implicitly assumed that the number of children and level of education do not violate the time constraints.

Maximization of the individual utility functions given these constraints lead to males' and females' notional demand for children,  $\tilde{n}_{1ik}$  and  $\tilde{n}_{2ik}$ . Negotiation leads to common effective demands  $\bar{n}_{1ik}$  and  $\bar{n}_{2ik}$  along with a level of government care  $\bar{G}_i$ . Taxes used to finance publicly provided old age care is paid by those at working ages and is denoted by  $\tau_i$ . Finally the individual effective demands for consumption, education,  $\bar{e}_{if}$  and  $\bar{e}_{im}$ , wealth accumulation  $W_i$ , and care of parents,  $\bar{E}_{if}$  and  $\bar{E}_{im}$ , are found by maximizing the couples common welfare function given the

common household constraints:

$$\begin{aligned}
c_{1i} &= \frac{1}{2} \left[ w_{1if} (T - \Gamma_i \cdot e_{if} - \varphi_{1if} \cdot \bar{n}_{1i}) + w_{1im} (T - \Gamma_i \cdot e_{im} - \varphi_{1im} \cdot \bar{n}_{1i}) - s_{1i} - 2\tau_{i+1} \right] \\
c_{2i} &= \frac{1}{2} \left[ w_{2if} (T - \varphi_{1if} \cdot \bar{n}_{2i} - \varphi_{2i} \cdot \bar{n}_{1i}) + w_{2im} (T - \varphi_{1im} \cdot \bar{n}_{2i} - \varphi_{2i} \cdot \bar{n}_{1i}) \right. \\
&\quad \left. - \pi_{i+2}^* \eta_{2i} (p_{E,i+2,f} E_{if} + p_{E,i+2,m} E_{im}) + (1 + r_{i+2}) s_{1i} - s_{2i} + 2B_{i,i-2} - 2\tau_{i+2} \right] \quad (6) \\
c_{3i} &= \frac{1}{2} \left[ (w_{3if} + w_{3im}) (T - \varphi_{2i} \cdot \bar{n}_{2i}) \right. \\
&\quad \left. - \pi_{i+3}^* \eta_{1i} (p_{E,i+3,f} E_{if} + p_{E,i+3,m} E_{im}) + (1 + r_{i+3}) s_{2i} - s_{3i} + 2B_{i,i-1} - 2\tau_{i+3} \right] \\
c_{4i} &= \frac{1}{2} (1 + r_{i+4}) s_{3i} - W_i
\end{aligned}$$

where  $c_{ji}$  is per person consumption in the household and  $s_{ji}$  is total savings (notice that savings are the only variables not being on a per person basis). This assumes that consumption is evenly shared between males and females in each period. The couple agrees on a common level of wealth accumulation, but different educational levels and different levels of care for parents.

Notice that in determining their notional demands (the individual budget constraint) individuals assume that they directly pay the cost of public care for their parents, while when determining their effective demands (the common household constraint) they pay taxes covering the care of the old during their work years. The cost of providing public old age care to a generation is thereby spread over the whole working population and not just the generation's children.

Figure 1 shows the interaction between generation  $i$ 's timeline and the timelines of the two parent generations  $i - 1$  and  $i - 2$ . The figure shows the number of children each of these two parent generations had, how these combine into generation  $i$ , and gives an illustration of when care of children, care of parents, education, and bequests occur during generation  $i$ 's lifetime.

For later use we define  $\varphi_{1ik}^*$  and  $\varphi_{2ik}^*$  as the discounted cost of having children at respectively age 1 and age 2 divided by the level of human capital,

$$\begin{aligned}
\varphi_{1ik}^* &= [w_{1ik} \varphi_{1k} + (1 + r_{i+2})^{-1} w_{2ik} \varphi_2] / h_{ik} \quad (7) \\
\varphi_{2ik}^* &= [(1 + r_{i+2})^{-1} w_{2ik} \varphi_{1k} + (1 + r_{i+2})^{-1} (1 + r_{i+3})^{-1} w_{3ik} \varphi_2] / h_{ik} .
\end{aligned}$$

The total discounted cost of having children at any age will be equal to the age specific time costs,  $\varphi_j$ , multiplied by the wage at the point in time the time cost is incurred.

Figure 1. Generation  $i$ 's timeline in relationship to parents' timelines\*

$t$	$i - 2$	$i - 1$	$i$	$i + 1$	$i + 2$	$i + 3$	$i + 4$
$i - 2$	----- ----- ----- ----- -----						
	0	1	2	3	4		
births		$n_{i-2,1}$	$n_{i-2,2}$				
$i - 1$	----- ----- ----- -----						
		0	1	2	3	4	
births			$n_{i-1,1}$	$n_{i-1,2}$			
$i$	----- ----- ----- -----						
			0	1	2	3	4
education				$e_i$			
births				$n_{i1}$	$n_{i2}$		
care of children				$\varphi_1 n_{i1}$	$\varphi_2 n_{i1}$		
					$\varphi_1 n_{i2}$	$\varphi_2 n_{i2}$	
care of parents					$\eta_{2i} E_i$	$\eta_{1i} E_i$	
bequests					$B_{i,i-2}$	$B_{i,i-1}$	$-W_i$

\* The subscript  $k$  has been dropped in the table

## 2.5 Bequests

The probability of females born at age  $i$  surviving until age 4 was earlier denoted by  $\pi_{i+4,f}$  and the male survival probability by  $\pi_{i+4,m}$ . If one or both parents die before reaching age 4, their unused savings (planned to be used as consumption at age 4) are given to their children in addition to the accumulated wealth the children were to receive anyway (bequests are shared out equally to all children). Letting  $\bar{c}_{ij}$  be the per person effective consumption (the agreed upon consumption level after the couple have negotiated about children) of males and females of generation  $i$  at age  $j$ , the total bequest per child given by those born at time  $i - 1$  will be

$$B_{i-1} = \frac{2\bar{W}_{i-1} + (1 - \pi_{i+3,f})\bar{c}_{4i-1} + (1 - \pi_{i+3,m})\bar{c}_{4i-1}}{\bar{n}_{1i-1} + \bar{n}_{2i-1}}$$

consisting of saved wealth and unused consumption due to the possibility of not reaching age 4. The total bequest received by individuals of generation  $i$  with young parents (born at time  $i - 1$ ),

$B_{i,i-1}$ , will be

$$B_{i,i-1} = \eta_{1i} \cdot B_{i-1} = 2\eta_{1i} \left( \frac{\bar{W}_{i-1} + (1 - \pi_{i+3}^*) 2\bar{c}_{4i-1}}{\bar{n}_{1i-1} + \bar{n}_{2i-1}} \right)$$

where  $\eta_{1i}$  is the proportion of generation  $i$  with younger parents leaving bequests when their children reach age 3,  $\eta_{ji} = \bar{n}_{i-2,j} / N_i$ , and where average longevity is defined as

$$\pi_{i+j}^* = \frac{1}{2} (\pi_{i+j,f} + \pi_{i+j,m}).$$

In the same manner the total bequest received by those with older parents (born at time  $i-2$ ),  $\hat{B}_{i,i-2}$ , will be

$$B_{i,i-2} = \eta_{2i} \cdot B_{i-2} = 2\eta_{2i} \left( \frac{\bar{W}_{i-2} + (1 - \pi_{i+2}^*) \bar{c}_{4i-2}}{\bar{n}_{1i-2} + \bar{n}_{2i-2}} \right),$$

where  $\eta_{2i}$  is the proportion of generation  $i$  that was born late and therefore have elderly parents who leave bequests when the children reach age 2.

Taking into account that those with old parents receive their bequest at age 2 and those with younger parents at age 3, the total discounted value of bequests received by each individual in generation  $i$  as a whole, denoted  $B_i^*$ , is given by

$$B_i^* = (1 + r_{i+2})^{-1} B_{i,i-2} + (1 + r_{i+2})^{-1} (1 + r_{i+3})^{-1} B_{i,i-1}. \quad (8)$$

The inclusion of mortality in the model makes it natural to include bequests (or at least involuntary bequests since individuals can die prematurely). Since bequests depend on one's parents income (which further depends on earlier generations income and so on in an infinite regress), they complicate the dynamic structure of the model. This is dealt with by assuming that individuals make some simplifying approximations (see assumption 2 in section 3.2) when finding the notional demands to be used in the negotiation phase. It is important to note that these approximations only directly affect notional demands, so that the budget constraints in effective demands are not violated.

## 2.6 Education and human capital

The human capital acquired through education by an individual of type  $k$  belonging to generation  $i$  is denoted  $h_{ik}$ . It accrues to the individual in the period the education is taken and is determined by the aggregate level of human capital in the previous period,  $\bar{h}_{i-1}$  and the amount of education,  $e_{ik}$ , the individual has taken. Letting  $\bar{e}_{ik}$  be the actual (effective) demand for education, the individuals' realized level of human capital will be

$$h_{ik} = (\bar{h}_{i-1})^{\theta_1} (\bar{e}_{ik})^{\theta_2}, \quad (9)$$

where  $\theta_1$  and  $\theta_2$  are parameters. Note that  $\bar{e}_{ik}$  is the effective realized demand for education, conditioned on the negotiated number of children, while the notional demand for education,  $\tilde{e}_{ik}$ , is based only on each individual's own desire for children.

The level of aggregate human capital for generation  $i$  is denoted  $\bar{h}_i$  and is determined by the previous level,  $\bar{h}_{i-1}$ , and the amount of education taken by women,  $\bar{e}_{if}$ , and men,  $\bar{e}_{im}$ ,

$$\bar{h}_i = (\bar{h}_{i-1})^{\theta_1} \left( \frac{\bar{e}_{if} + \bar{e}_{im}}{2} \right)^{\theta_2}. \quad (10)$$

In the following, we will distinguish between human capital, which in our use of the term only takes education into account, and skills, which also take experience into account. The skill of an individual of type  $k$  of generation  $i$  at age  $j$ ,  $S_{jik}$ , is assumed to be the product of an age specific experience parameter  $\lambda_j$  and the level of human capital. An individual of generation  $i = t$  is assumed to have the following skill levels at different ages (at time  $t = i + j$ ),

$$\begin{aligned} S_{1ik} &= \lambda_1 h_{ik} = \lambda_1 (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} \\ S_{2ik} &= \lambda_2 h_{ik} = \lambda_2 (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} \\ S_{3ik} &= \lambda_3 h_{ik} = \lambda_3 (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}. \end{aligned} \quad (11)$$

If the discounted value of these experience parameters increases with age for a generation born at time  $i$ , we have:

$$\lambda_1 < (1 + r_{i+2})^{-1} \lambda_2 < (1 + r_{i+3})^{-2} \lambda_3. \quad (12)$$

The number of persons of age  $j$  at time  $t$  will be  $N_{t-j}$ . Letting  $l_{jik}$  be the amount of labor supplied by individuals of type  $k$  belonging to cohort  $i$  at age  $j$ , the total quantity of efficiency-labor employed in production at time  $t$ ,  $H_t$ , is assumed to be given by:

$$H_t = \sum_{j=1}^3 N_{t-j} \sum_{k=f,m} l_{j,t-j,k} S_{j,t-j,k} - \sum_{j=1}^3 \sum_{k=f,m} L_{j,t-j,k}^G \quad (13)$$

where  $\sum_{j=1}^3 \sum_{k=f,m} L_{j,t-j,k}^G$  is the amount of labor used by the government in providing old care. In the above skills are assumed to be perfect substitutes in production. Males and females work in general different hours due to time used on education and caring for children, but at age 3 they work equal amount of hours.

## 2.7 Production and factor prices

Production  $Y_t$  occurs within a period according to a standard one-sector production function  $F$  that exhibits constant returns to scale and is subject to exogenous technological progress through a technological coefficient  $\psi_t$ , reflecting labor augmenting technological change at time  $t$ . The

technological coefficient is an externality that is not taken into account by the firms or individuals when they maximize profits and utility. Letting  $K_t$  be capital, output at time  $t$  is assumed to be given by

$$Y_t = F(K_t, \psi_t H_t) = AK_t^\mu (\psi_t H_t)^{1-\mu} = \psi_t H_t f(k_t), \quad (14)$$

where  $k_t = K_t/(\psi_t H_t)$ ,  $f(k_t) = Ak_t^\mu$ , and  $A$  and  $\mu$  are parameters.

A small open economy is assumed, so that the real interest rate,  $r_t$ , is given exogenously by international capital markets. The solution to the firms' optimization problem sets factor costs equal to their marginal productivity. For capital this gives

$$r_t = \partial Y_t / \partial K_t = f'(k_t), \quad (15)$$

which implies that the capital ratio  $k_t$  is determined exogenously in the capital markets,  $k_t = \left(\frac{r_t}{\mu A}\right)^{1/(\mu-1)}$ .

Profit maximization implies that the marginal productivity of efficiency-labor  $H_t$  is given by  $\partial Y_t / \partial H_t = \psi_t [f(k_t) - k_t f'(k_t)]$ . At the disaggregated level, marginal productivity determines the following sequence of per capita wages for generation  $i$  (born at time  $t = i$ )

$$\begin{aligned} w_{1ik} &= \psi_{i+1} \bar{w}(r_{i+1}) \lambda_1 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, \\ w_{2ik} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, \\ w_{3ik} &= \psi_{i+3} \bar{w}(r_{i+3}) \lambda_3 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, \end{aligned} \quad (16)$$

where  $w_{jik} = \frac{1}{N_{t-j}} \cdot (\partial Y_t / \partial H_t) (\partial H_t / \partial l_{j,t-j,k})$  for  $t = i + j$  is the per capita wage for individuals of generation  $i$  at age  $j$  and  $\bar{w}(r_t)$  is defined by the function

$$\bar{w}(r_t) = f(k_t(r_t)) - k_t(r_t) f'(k_t(r_t)) = (1 - \mu) A (k_t(r_t))^\mu = (1 - \mu) A \left(\frac{r_t}{\mu A}\right)^{\mu/(\mu-1)}.$$

To see how this looks at a point in time, consider for example time  $t = i + 2$  in which firms employ the three cohorts  $i - 1$ ,  $i$  and  $i + 1$ . They pay different wages to the three cohorts based on the cohort's different levels of experience and human capital in the following manner:

$$\begin{aligned} w_{1,i+1,k} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_1 \cdot (\bar{h}_i)^{\theta_1} (e_{i+1,k})^{\theta_2} \\ w_{2ik} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, \\ w_{3,i-1,k} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_3 \cdot (\bar{h}_{i-2})^{\theta_1} (e_{i-1,k})^{\theta_2}. \end{aligned}$$

Government care to generation  $i$  (at time  $i + 4$ ) is provided through a linear production function

$$\bar{G}_i = \min \left[ \frac{L_{1,i+3,f}^G}{\mu_G}, \frac{L_{1,i+3,m}^G}{\mu_G}, \frac{L_{2,i+2,f}^G}{\mu_G}, \frac{L_{2,i+2,m}^G}{\mu_G}, \frac{L_{3,i+1,f}^G}{\mu_G}, \frac{L_{3,i+1,m}^G}{\mu_G} \right]$$

so that employment is given by

$$L_{j,i+j,k}^G = \mu_G \bar{G}_i, \quad j = 1, 2, 3, \quad k = f, m.$$

To be able to entice individuals to work providing old age care, they must receive the same wage as in the normal production sector. In addition it is assumed that there is a fixed cost component  $p_{0G}$  which does not change over time. We thereby have that the price of a unit government care for the elderly is

$$p_{G^{i+4}}^* = p_{0G} + (w_{1,i+3,f} + w_{1,i+3,m} + w_{1,i+2,f} + w_{1,i+2,m} + w_{1,i+1,f} + w_{1,i+1,m}) \mu_G.$$

The fixed cost component can play a large effect at low income levels, becoming less and less important as incomes increase. It can thereby explain low initial levels of government care.

Government provided care for the elderly is covered by the lump-sum tax

$$\tau_{i+4} = \frac{p_{G^{i+4}}^* \cdot \pi_{i+4}^* N_i \cdot \bar{G}_i}{N_{i+1} + N_{i+2} + N_{i+3}}.$$

### 3 Notional demand for children

#### 3.1 Expectations

So as to ensure closed form solutions to the agents' maximization problem (important when considering the negotiation stage of the decision process), simplifying assumptions are made about the individuals expectations about the behavior of parents regarding bequests and the behavior of children regarding care when reaching old age.

*Assumption 1 (Self-referring expectations concerning children's behavior).* Individuals behave as if  $E_{ik}^* = E_{ik}$  and  $G_{ik}^* = G_{ik}$ . Individuals expect to receive from their children the same care they give their own parents.

Assumption 1 states that the subutility of *receiving* care from ones children,  $u_{E_i}$ , is transformed to a function of the amount of care *given* to ones parents,  $E_i$  (in addition to the number of children born to the individual). This is based on agents believing there is a social contract stipulating that they will receive the same amount of care from each of their children as they give their parents.

Define  $R_{ik}^M$  as the notional monetary income (labor income plus bequests minus taxes) of an individual of type  $k$  in generation  $i$  and  $\bar{R}_i^M$  as a couples corresponding total effective monetary income.



*Assumption 2 (Rule-of-thumb approximations concerning parents' behavior regarding bequests).*

When deriving their notional demand for children,  $\tilde{n}_{1ik}$  and  $\tilde{n}_{2ik}$ , individuals of type  $k$  and generation  $i$  assume that the bequests they will receive from generation  $j$  are given by  $\hat{B}_{i,i-j,k}$ , which approximates actual bequests  $B_{i,i-j}$  by replacing in the function for bequests the human capital levels of ones' parents,

$$h_{i-j,f} \text{ and } h_{i-j,m} \text{ for } k = f, m \text{ and } j = 1, 2,$$

by one's own notional choice of human capital level,

$$h_i = (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}.$$

In addition individuals assume that the notional monetary income of their parents,  $R_{i-j,k}^M$ , is equal to the effective monetary income of their parents,  $\bar{R}_{i-j}^M$ , so that

$$R_{i-1,k}^M \cong \bar{R}_{i-1}^M \text{ and } R_{i-2,k}^M \cong \bar{R}_{i-2}^M.$$

Assumption 2 states that the individuals use some rules of thumb to estimate the amount of bequests they will receive. As long as bequests are a fairly small part of the present value of total income,

$$w_{1ik}T + (1 + r_{i+2})^{-1}w_{2ik}T + (1 + r_{i+2})^{-1}(1 + r_{i+3})^{-1}w_{3ik}T \gg B_i^* \cong \hat{B}_{ik}^*,$$

these approximations should have little effect on each generation's demand (keeping in mind that one cannot in general rule out that small changes in the demand of each generation can have significant effects on the dynamics of the economy). The approximations are such that bequests do not introduce extra dynamics to the model, retaining it's simplicity.

As is usual in endogenous growth models, it is assumed that there are externalities in the model. In addition to the normal assumption that agents do not take into account their influence on technological progress, it is also assumed that they do not take into account their influence on the price of care for the elderly.

*Assumption 3 (Externalities concerning technological progress, price of care for the elderly and taxes).* Firms and individuals plan as if technological progress,  $\psi$ , is exogenous, and individuals plan as if the prices of care for the elderly,  $p_{Ei}$  and  $p_{Gi}$ , and taxes  $\tau_i$  are exogenous.

### 3.2 Notional demand for children

Each individual maximizes utility to find the notional demands for children that will play an important part in the negotiation phase.

**Lemma 1 (Notional demand for children)** *Maximizing (3) with respect to  $s_{1ik}$ ,  $s_{2ik}$ ,  $s_{3ik}$ ,  $W_{ik}$ ,  $E_{ik}$ ,  $G_{ik}$ ,  $n_{1ik}$ ,  $n_{2ik}$ , and  $e_{ik}$  subject to (5), (16), (4), and (1), along with assumptions 1-3 leads to the following demand equations for children:*

$$\tilde{n}_{1ik} = \frac{1}{2\varphi_{1ik}^*} \cdot \frac{\Lambda_{Nik} + \Lambda_{Eik}}{1 + \theta_2(1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right], \quad k = f, m \quad (17)$$

$$\tilde{n}_{2ik} = \frac{1}{2\varphi_{2ik}^*} \cdot \frac{\Lambda_{Nik} + \Lambda_{Eik}}{1 + \theta_2(1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right], \quad k = f, m \quad (18)$$

where  $\tilde{e}_{ik}$  is the notional demand for education and where

$$\begin{aligned} d_{kj} &= (1 + r_k) \prod_{l=k}^j (1 + r_l)^{-1} \\ \lambda_{1i}^* &= \psi_{i+1} \bar{w}(r_{i+1}) \cdot \lambda_1 \\ \lambda_{2i}^* &= d_{i+1, i+2} \psi_{i+2} \bar{w}(r_{i+2}) \cdot \lambda_2 \\ \lambda_{3i}^* &= d_{i+1, i+3} \psi_{i+3} \bar{w}(r_{i+3}) \cdot \lambda_3 \\ \varphi_{1ik}^* &= \lambda_{1i}^* \cdot \varphi_{1ik} + \lambda_{2i}^* \cdot \varphi_{2i} \\ \varphi_{2ik}^* &= \lambda_{2i}^* \cdot \varphi_{1ik} + \lambda_{3i}^* \cdot \varphi_{2i} \\ T^* &= \lambda_{1i}^* \cdot T + \lambda_{2i}^* \cdot T + \lambda_{3i}^* \cdot T \\ \tilde{h}_{ik} &= (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2} \\ \Lambda_{Nik} &= \alpha_0 / \Lambda_{ik}^* \\ \Lambda_{Eik} &= \pi_{i+4, k} \alpha_{Ei} / \Lambda_{ik}^* \\ \Lambda_{ik}^* &= 1 + \beta + \beta^2 + \pi_{i+4, k} \beta^3 + \pi_{i+4, k} \alpha_{Ei} + \alpha_W + \alpha_0 + \pi_{i+4, k} \alpha_{Ei}. \end{aligned}$$

**Proof.** See appendix A. ■

In general, bequests,  $B_{ik}^*$ , will depend on all previous variables. This poses no problems for the above derivation of notional demand for children, since they can be considered predetermined, but will lead to complicated dynamics. Using assumption 2 to substitute  $\hat{B}_{ik}^*$  for  $B_{ik}^*$  simplifies the model greatly, as reflected in lemma 2.

**Lemma 2 (Expected bequests received)** *Under assumption 2, generation  $i$ 's expected received bequests divided by their notional human capital level,  $\frac{\hat{B}_{ik}^*}{(\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2}}$ , will only depend on parameters in the model with*

$$\hat{B}_{i-j, f}^* / (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{if})^{\theta_2} = \hat{B}_{i-j, m}^* / (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{im})^{\theta_2}.$$

**Proof.** See appendix D. ■

The expression on the right hand side of the notional demand equations (17) and (18) can be written as individual  $k$ 's total expected discounted income over his or her life,

$$\frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2} \cdot T_i^* + (1 + \theta_2) \hat{B}_{ik}^* \right] = \sum_{j=1}^3 w_{jik} \cdot T - w_{1ik} \Gamma_i e_{ik} + \hat{B}_{ik}^*,$$

where the endogeneity of education has been taken into account. Rearranging equations (17) and (18) and then inserting from (7) to get

$$(w_{1ik} \varphi_{1k} + (1 + r_{i+2})^{-1} w_{2ik} \varphi_2) \cdot \tilde{n}_{1ik} = \frac{\Lambda_{Nik} + \Lambda_{Eik}}{2} \cdot \left( \sum_{j=1}^3 w_{jik} \cdot T - w_{1ik} \Gamma_i e_{ik} + \hat{B}_{ik}^* \right)$$

and

$$\begin{aligned} ((1 + r_{i+2})^{-1} w_{2ik} \varphi_{1k} + (1 + r_{i+2})^{-1} (1 + r_{i+3})^{-1} w_{3ik} \varphi_2) \cdot \tilde{n}_{2ik} \\ = \frac{\Lambda_{Nik} + \Lambda_{Eik}}{2} \cdot \left( \sum_{j=1}^3 w_{jik} \cdot T - w_{1ik} \Gamma_i e_{ik} + \hat{B}_{ik}^* \right), \end{aligned}$$

it can be seen that the notional demand equations for children imply that the marginal cost of having children must equal the marginal benefit in money terms.

The demand equations for children, (17) and (18), imply that increases in the costs of having children,  $\varphi_{1ik}^*$  and  $\varphi_{2ik}^*$ , will decrease notional demand, while increases in discounted lifetime income,  $T_i^* + (1 + \theta_2) \hat{B}_{ik}^* / (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2}$ , will increase it.

Differences in the discounted costs of having children early,  $\varphi_{1ik}^*$ , and having them late,  $\varphi_{2ik}^*$ , determine the distribution between early born and late born children. The fact that women experience higher costs than men ( $\varphi_{jif}^* > \varphi_{jim}^*$ ), will in isolation lead to women wanting fewer children than men. On the other hand, as will be seen later, the parameters of the utility function (especially if the relative size of  $\alpha_{Ei}$  is large and  $\pi_{i+4,f} > \pi_{i+4,m}$ ) can counteract this effect. If  $\alpha_{Ei}$  is small enough,  $\varphi_{1if}$  is larger than  $\varphi_{1im}$  and  $\pi_{i+4,f}$  is larger than  $\pi_{i+4,m}$ , then women will always want less children than men,  $\tilde{n}_{jif} < \tilde{n}_{jim}$ .

Spacing is defined as the difference between the number of children born late and the number born early,  $\bar{n}_{2i} - \bar{n}_{1i}$ . An increase in late births compared to early births will according to this definition increase spacing.

**Proposition 1** *More children will be born early than late if  $\varphi_{2ik}^* > \varphi_{1ik}^*$ , which will always be the case if discounted wages increase over time,*

$$w_{1ik} < (1 + r_{i+2})^{-1} w_{2ik} < (1 + r_{i+2})^{-1} (1 + r_{i+3})^{-1} w_{3ik},$$

**Proof.** Inserting for  $\varphi_{2ik}^*$  and  $\varphi_{1ik}^*$  in  $\varphi_{2ik}^* > \varphi_{1ik}^*$  and multiplying by  $h_{ik}$  leads to  $(d_{i+1,i+2}w_{2ik} - w_{1ik}) \cdot \varphi_{1ik} > (d_{i+1,i+2}w_{2ik} - d_{i+1,i+3}w_{3ik}) \cdot \varphi_{2i}$ . ■

One might want to note that if more children are born early than late,  $\bar{n}_{1i} > \bar{n}_{2i}$ , then an increase in spacing will always increase average age when giving birth.

## 4 Effective demands

### 4.1 Effective demand for children as the outcome of bargaining

It is assumed that the effective (actual) demand for children is the outcome of negotiation between the sexes. As mentioned in the introduction, negotiation between the sexes is based on the notional demands for children found in the previous section. The couple negotiate on the basis of these notional demands, agreeing on the number of children they wish to have; their collective effective demand for children.

Individuals wish to minimize the distance between the agreed upon number of children born at time  $j$  and their notional demand,  $\tilde{n}_{jik}$ :

$$\min_{n_{ji}} (\tilde{n}_{jik} - n_{ji})^2, \quad j = 1, 2, \quad k = f, m.$$

The Nash product gives the  $n_{ji}$  that is the solution to the bargaining problem

$$\min \left( (\tilde{n}_{jif} - n_{ji})^2 \right)^{\Phi_{fi}^*} \left( (\tilde{n}_{jim} - n_{ji})^2 \right)^{\Phi_{mi}^*}, \quad j = 1, 2,$$

where  $\Phi_{mi}^*$  is a parameter reflecting the negotiation strength of males and  $\Phi_{fi}^*$  the negotiation strength of females.

The effective demand for children born early,  $\bar{n}_{1i}$ , is then given by

$$\bar{n}_{1i} = \Phi_i \tilde{n}_{1if} + (1 - \Phi_i) \tilde{n}_{1im}$$

and for children born later in life

$$\bar{n}_{2i} = \Phi_i \tilde{n}_{2if} + (1 - \Phi_i) \tilde{n}_{2im}$$

where  $\Phi_i = \Phi_{fi}^* / (\Phi_{fi}^* + \Phi_{mi}^*)$ . The realized number of children,  $\bar{n}_{ji}$ , is the weighted average of the desires of the male and the female weighted with their relative bargaining power.

**Lemma 3 (Effective demand for children)** *The realized number of births determined by Nash product is given by*

$$\bar{n}_{1i} = \Omega_{1i} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{if}^*}{(\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{if})^{\theta_2}} \right] \quad (19)$$

$$\bar{n}_{2i} = \Omega_{1i} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{if}^*}{(\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{if})^{\theta_2}} \right] \quad (20)$$

with

$$\begin{aligned} \Omega_{1i} &= \frac{\Lambda_{Nif} + \Lambda_{Eif}}{2\varphi_{1if}^*} \cdot \frac{\Phi_i}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} + \frac{\Lambda_{Nim} + \Lambda_{Eim}}{2\varphi_{1im}^*} \cdot \frac{1 - \Phi_i}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})} \\ \Omega_{2i} &= \frac{\Lambda_{Nif} + \Lambda_{Eif}}{2\varphi_{2if}^*} \cdot \frac{\Phi_i}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} + \frac{\Lambda_{Nim} + \Lambda_{Eim}}{2\varphi_{2im}^*} \cdot \frac{1 - \Phi_i}{1 + \theta_2 (1 - \Lambda_{Nim} + \Lambda_{Eim})}. \end{aligned}$$

where by lemma 2

$$\frac{\hat{B}_{if}^*}{(\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{if})^{\theta_2}} = \frac{\hat{B}_{im}^*}{(\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{im})^{\theta_2}}.$$

If women want less children than men,  $\tilde{n}_{jif} < \tilde{n}_{jim}$  (if  $\alpha_{Ei}$  is small enough,  $\varphi_{1if}$  is larger than  $\varphi_{1im}$  and  $\pi_{i+4,f}$  is larger than  $\pi_{i+4,m}$ ), then an increase in the relative negotiation strength of women,  $\Phi_i$ , will lead to there being born less children.

## 4.2 Government provided old age care

At time  $i + 4$  (the time generation  $i$  goes into retirement) there are four generations of voting age (ages 1 to 4). Their notional demand for giving government old age care is given by

$$\tilde{G}_{ik} = \frac{1}{p_{Gi}^*} \cdot \frac{\Lambda_{Gik}}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right] \tilde{h}_{ik}.$$

It is assumed that the social contract between generation is such that they receive an average of male and female notional demands,

$$\bar{G}_i = \frac{1}{2} \left( \frac{\tilde{G}_{if}}{\tilde{h}_{if}} + \frac{\tilde{G}_{im}}{\tilde{h}_{im}} \right) \cdot \bar{h}_i$$

to be paid for through taxes.

## 4.3 The household welfare function and the effective demand for education

Inserting the above choice of number of children into the individual utility functions gives us the following utility functions for males and females

$$\begin{aligned} U_{if} &= \ln c_{1if} + \beta \ln c_{2if} + \beta^2 \ln c_{3if} + \pi_{i+4,f} \beta^3 \ln c_{4if} + \alpha_W \ln W_{if} + \pi_{i+4,f} \alpha_{Ei} \cdot \ln E_{if} \\ &\quad + \frac{\alpha_0 + \pi_{i+4,f} \cdot \alpha_{Ei}}{2} (\ln \bar{n}_{1i} + \ln \bar{n}_{2i}) + \pi_{i+4,f} \cdot \alpha_G \ln (1 + \bar{G}_i) \end{aligned}$$

$$\begin{aligned} U_{im} &= \ln c_{1im} + \beta \ln c_{2im} + \beta^2 \ln c_{3im} + \pi_{i+4,m} \beta^3 \ln c_{4im} + \alpha_W \ln W_{im} + \pi_{i+4,m} \alpha_{Ei} \cdot \ln E_{im} \\ &\quad + \frac{\alpha_0 + \pi_{i+4,m} \cdot \alpha_{Ei}}{2} (\ln \bar{n}_{1i} + \ln \bar{n}_{2i}) + \pi_{i+4,m} \cdot \alpha_G \ln (1 + \bar{G}_i), \end{aligned}$$

Assuming equal levels of consumption and equal wealth accumulation for males and females,

$$\begin{aligned} c_{1if} &= c_{1im} = c_{1i} & c_{3if} &= c_{3im} = c_{3i} \\ c_{2if} &= c_{2im} = c_{2i} & c_{4if} &= c_{4im} = c_{4i} \\ W_{if} &= W_{im} = W_i \end{aligned}$$

( $c_{ji}$  are per person consumption), the household welfare function is defined as the average of the male and female utility functions:

$$\begin{aligned} \bar{U}_i &= \frac{1}{2}U_{if} + \frac{1}{2}U_{im} = \ln c_{1i} + \beta \ln c_{2i} + \beta^2 \ln c_{3i} + \pi_{i+4}^* \beta^3 \ln c_{4i} + \alpha_W \ln W_i \\ &\quad + \frac{1}{2} \pi_{i+4,f} \alpha_{Ei} \cdot \ln E_{if} + \frac{1}{2} \pi_{i+4,m} \alpha_{Ei} \cdot \ln E_{im} \\ &\quad + (\alpha_0 + \pi_{i+4}^* \cdot \alpha_{Ei}) (\ln \bar{n}_{1i} + \ln \bar{n}_{2i}) + \pi_{i+4}^* \cdot \alpha_G \ln (1 + \bar{G}_{ik}) \end{aligned} \quad (21)$$

**Lemma 4 (Effective demand for education)** *Maximizing (21) with respect to  $s_{1i}$ ,  $s_{2i}$ ,  $s_{3i}$ ,  $e_{if}$ ,  $e_{im}$ ,  $W_i$ ,  $E_{if}$ , and  $E_{im}$  subject to (6) leads to the following effective demand for education:*

$$\bar{e}_{if} = \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \cdot \Gamma_i} [T_i^* - \varphi_{1if}^* \cdot \bar{n}_{1i} - \varphi_{2if}^* \cdot \bar{n}_{2i}] \quad (22)$$

and

$$\bar{e}_{im} = \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \cdot \Gamma_i} [T_i^* - \varphi_{1im}^* \cdot \bar{n}_{1i} - \varphi_{2im}^* \cdot \bar{n}_{2i}], \quad (23)$$

**Proof.** See appendix B. ■

**Lemma 5 (Actual bequests given)** *Individuals in generation  $i - j$  leave behind actual bequests to generation  $i$  of  $B_{i,i-j}$ ,*

$$\begin{aligned} B_{i,i-j} &= \eta_{ji} \cdot B_{i-j} \\ &= \eta_{ji} \frac{\Omega_{0,i-j}}{\Omega_{1,i-j} + \Omega_{2,i-j}} \cdot \bar{R}_{i-j}^M \cdot \left( \frac{1}{\tilde{h}_{i-j,f}} R_{i-j,f}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f}} + \theta_2 \right) \right. \\ &\quad \left. + \frac{1}{\tilde{h}_{i-j,m}} R_{i-j,m}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m}} + \theta_2 \right) \right)^{-1} \end{aligned} \quad (D.1)$$

where

$$\begin{aligned}\Omega_{0,i-j} &= \frac{1}{d_{i-j+1,i-j+4}} \cdot \frac{(1 - \pi_{i-j+4}^*) \pi_{i-j+4}^* \beta^3 + \alpha_W}{1 + \beta + \beta^2 + \pi_{i-j+4}^* \beta^3 + \pi_{i-j+4}^* \alpha_{Ei} + \alpha_W} \\ \Omega_{1,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{1,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{\Lambda_{N,i-j,m} + \Lambda_{E,i-j,m}}{2\varphi_{1,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})} \\ \Omega_{2,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{2,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{1}{2\varphi_{2,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})}\end{aligned}$$

**Proof.** See appendix C. ■

## 5 Dynamics

If  $\alpha_{Ei}$  is constant (or equivalently that  $\bar{G}_i$  is greater than  $\hat{G}$ ) and the level of productivity and the interest rate do not change, then the dynamics of the model are very simple. The main dynamics are then provided by the development in human capital given by a first order difference equation which is stable if  $\theta_1 + \theta_2 \Gamma_{i1} < 1$ . The dynamics of the model become somewhat more complicated while government care for the elderly still influences the parameter  $\alpha_{Ei}$ , which with a certain lag affects human capital, population growth and new desired levels of government care.

The effective demand for government expenditure per unit human capital was earlier found to be

$$\frac{\bar{G}_i}{\bar{h}_i} = \frac{1}{p_{G_i}^*} \cdot \left( \frac{1}{2} \sum_{k=f,m} \frac{\Lambda_{Gik}}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \right) \cdot \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\bar{h}_{ik}} \right].$$

By differentiating this with respect to  $\alpha_{Ei}$  it can be seen that for increasing  $\bar{h}$ ,  $\bar{G}$  will be always be increasing as long as wealth accumulation and consumption have a greater weight in the utility function than children:  $1 + \beta + \beta^2 + \pi_{i+4,k} \beta^3 + \alpha_W > \alpha_0$ . If we, for example, take as a starting point a situation where the level of human capital,  $\bar{h}$ , is growing, growth in human capital levels and thereby in income will have a first order effect of increasing government expenditures on old age care,  $\bar{G}_i$ . The increase in  $\bar{G}_i$  will decrease the utility of private care,  $\alpha_{Ei+4}$ , which after a four period lag will have a second order effect of further increasing  $\bar{G}_i$ .

Earlier we defined the utility parameter for private old age care as

$$\alpha_{Ei} = \begin{cases} \xi_1 + \xi_2 \cdot (1 + \bar{G}_{i-4})^{-1} & \text{if } 0 < \bar{G}_{i-4} < \hat{G} \\ \xi_1 & \text{if } \bar{G}_{i-4} \geq \hat{G} \end{cases}$$

Define

$$\left(\frac{G}{h}\right)_i^{\min} = \frac{\bar{G}_i}{\bar{h}_i} \Big|_{\alpha_{Ei}=\xi_1}$$

where  $\bar{G}_i/\bar{h}_i$  is conditioned on  $\alpha_{Ei}$  having its minimum value and

$$\left(\frac{G}{h}\right)_i^{\max} = \frac{\bar{G}_i}{\bar{h}_i} \Big|_{\alpha_{Ei}=\xi_1+\xi_2}$$

where  $\bar{G}_i/\bar{h}_i$  is conditioned on  $\alpha_{Ei}$  having its maximum value.

Further define

$$h_i^{\hat{G}} = \frac{\left(\frac{G}{h}\right)_i^{\min}}{\hat{G}}.$$

At all levels equal to or greater than  $h_i^{G^*}$  we have that  $\alpha_{Ei}$  is equal to  $\xi_1$ .

From before we know that  $\Omega_{1i}$ ,  $\Omega_{2i}$ , and  $\frac{\hat{B}_{ik}^*}{\hat{h}_{ik}}$  depend on values of  $\alpha_E$ , which we denote as follows:

$$\Omega_{1i}(\alpha_{Ei}), \quad \Omega_{2i}(\alpha_{Ei}) \quad \text{and} \quad \frac{\hat{B}_{ik}^*}{\hat{h}_{ik}}(\alpha_{Ei-1}, \alpha_{Ei-2}).$$

Inserting for

$$\Gamma_i = \exp(-\Gamma_{i0}^*) \cdot (h_{i-1})^{-\Gamma_{i1}^*}$$

$$\begin{aligned} \Gamma_{i1}^* &> 0 \\ \Gamma_{i1}^* &> \frac{1-\theta_1}{\theta_2} \\ \theta_2 \Gamma_{i1}^* + \theta_1 &> 1 \end{aligned}$$

we get

$$\begin{aligned} \log h_i &= \theta_1 \log h_{i-1} + \theta_2 \log \left( \frac{\theta_2}{(1+\theta_2) \lambda_{1i}^*} \right) \\ &- \theta_2 \log \left[ \exp(-\Gamma_{i0}^*) \cdot (h_{i-1})^{-\Gamma_{i1}^*} \right] + \theta_2 \log \left[ T_i^* - \frac{\frac{\varphi_{1if}^* + \varphi_{1im}^*}{2} \Omega_{1i} + \frac{\varphi_{2if}^* + \varphi_{2im}^*}{2} \Omega_{2i}}{2} \cdot \left( T_i^* + (1+\theta_2) \frac{\hat{B}_{ik}^*}{\hat{h}_{ik}} \right) \right] \end{aligned}$$

Inserting the equations for education (22) and (22) into

$$h_i = (\bar{h}_{i-1})^{\theta_1} \left( \frac{\bar{e}_{if} + \bar{e}_{im}}{2} \right)^{\theta_2}$$

gives the equation

$$\begin{aligned} \log h_i &= (\theta_1 + \theta_2 \Gamma_{i1}) \log h_{i-1} + \theta_2 \log \left( \frac{\theta_2}{(1+\theta_2) \lambda_{1i}^* \exp(\Gamma_{i0})} \right) \\ &+ \theta_2 \log \left[ T_i^* - \frac{\frac{\varphi_{1if}^* + \varphi_{1im}^*}{2} \Omega_{1i} + \frac{\varphi_{2if}^* + \varphi_{2im}^*}{2} \Omega_{2i}}{2} \cdot \left( T_i^* + (1+\theta_2) \frac{\hat{B}_{ik}^*}{\hat{h}_{ik}} \right) \right] \end{aligned}$$



describing the dynamics of human capital development .The level of human capital in always growing if  $\log h_i - \log h_{i-1} > 0$ , or equivalently

$$\theta_2 \log \left( \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \exp(-\Gamma_{i0}^*)} \right) + \theta_2 \log \left[ T_i^* - \frac{\frac{\varphi_{1if}^* + \varphi_{1im}^* \Omega_{1i}}{2} + \frac{\varphi_{2if}^* + \varphi_{2im}^* \Omega_{2i}}{2}}{2} \cdot \left( T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\bar{h}_{ik}} \right) \right] > (1 - \theta_1 - \theta_2 \Gamma_{i1}^*) \log h_{i-1}$$

Define

$$H_i(x, y, z) = \theta_2 \left\{ \log \left( \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \exp(-\Gamma_{i0}^*)} \right) + \log \left[ T_i^* - \frac{\frac{\varphi_{1if}^* + \varphi_{1im}^* \Omega_{1i}(x)}{2} + \frac{\varphi_{2if}^* + \varphi_{2im}^* \Omega_{2i}(x)}{2}}{2} \cdot \left( T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\bar{h}_{ik}}(y, z) \right) \right] \right\}$$

At time  $i$  this will be equal to

$$H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2}),$$

so that the level of human capital is always growing if

$$\frac{H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2})}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*} > \log h_{i-1}$$

Equilibrium will occur when

$$\frac{H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2})}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*} = \log h_{i-1}.$$

Assuming that equilibrium occurs after  $G$  has become greater than  $\hat{G}$  so that

$$\alpha_{Ei} = \xi_1,$$

let equilibrium  $H_i$  be defined as

$$H_i^{\text{Eq}} = H_i(\xi_1, \xi_1, \xi_1).$$

Above we defined  $h_i^{G^*}$  so that for values of  $h$  greater than this we have that  $\alpha_{Ei}$  is equal to  $\xi_1$  and

$$H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2}) = H_i^{\text{Eq}}.$$

Finally also define

$$H_i^{\text{min}} = H_i(\xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2),$$

which is the smallest possible  $H_i$ .

**Proposition 2** *If  $\hat{G}$  is chosen so that  $\log h_i^{\hat{G}} < \frac{H_i^{\text{min}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}$ , then human capital  $\bar{h}_i$  and government expenditures on old age care,  $\bar{G}_i$ , will be increasing as long as  $\bar{h}_i$  is below the equilibrium level  $\frac{H_i^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}$ . imply increasing levels of government old age care. The equilibrium is stable as long as  $(\theta_1 + \theta_2 \Gamma_{i1}^*) < 1$ .*

**Proof.** The level of human capital is increasing for all

$$\log h_i < \frac{H_i^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}$$

if

$$\log h_i^{G^*} < \frac{H_i^{\text{min}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*} \quad \left( < \frac{H_i^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*} \right)$$

If  $\bar{h}_i < h_i^{G^*}$  the level of human capital is always increasing because

$$\log h_i < \frac{H_i^{\text{min}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}.$$

At higher levels than  $h_i^{G^*}$  we have that  $H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2})$  becomes constant. The general solution to the first order difference equation

$$\begin{aligned} \log h_i &= (\theta_1 + \theta_2 \Gamma_{i1}^*) \log h_{i-1} + H_i(\alpha_{Ei}, \alpha_{Ei-1}, \alpha_{Ei-2}) \\ &= (\theta_1 + \theta_2 \Gamma_{i1}^*) \log h_{i-1} + H_i^{\text{Eq}} \end{aligned}$$

then becomes

$$\log h_i = (\theta_1 + \theta_2 \Gamma_{i1}^*)^{i-j} \left( \log h_j - \frac{H_j^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{j1}^*} \right) + \frac{H_j^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{j1}^*}$$

( $j$  being a time at which  $\alpha_{Ei}$  equals  $\xi_1$ ) with a stable equilibrium at  $h_*$  such that

$$\log h_* = \frac{H_i^{\text{Eq}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}.$$

■

The assumption  $\log h_i^{G^*} < \frac{H_i^{\text{min}}}{1 - \theta_1 - \theta_2 \Gamma_{i1}^*}$  ensures that decreases in  $\alpha_{Ei}$  die out before  $\bar{h}_i$  becomes so close to its equilibrium level that changes in  $\alpha_{Ei}$  could lead to a decline in  $\bar{h}_i$ .

Notice that  $H_i$  is a function of all the parameters in the model,

$$\begin{aligned} \exp(H_i) &= H(\theta_2, \Gamma_{i0}, \lambda_1, \lambda_2, \lambda_3, T, \psi_{i+1}, \psi_{i+2}, \psi_{i+3}, r_{i+1}, r_{i+2}, r_{i+3}, \varphi_{1i-2f}, \\ &\quad \varphi_{1i-1f}, \varphi_{1if}, \varphi_{1i-2m}, \varphi_{1i-1m}, \varphi_{1im}, \varphi_{2i-2}, \varphi_{2i-1}, \varphi_{2i}, \pi_{i+2,f}, \pi_{i+3,f}, \pi_{i+4,f}, \\ &\quad \pi_{i+2,m}, \pi_{i+3,m}, \pi_{i+4,m}, \Phi_{i-2}, \Phi_{i-1}, \Phi_i, \beta, \alpha_W, \alpha_{Ei-2}, \alpha_{Ei-1}, \alpha_{Ei}, \alpha_0) \\ &= \left\{ \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \cdot \exp(\Gamma_{i0})} \left[ T_i^* - \frac{\varphi_{1if}^* + \varphi_{1im}^* \Omega_{1i}}{2} + \frac{\varphi_{2if}^* + \varphi_{2im}^* \Omega_{2i}}{2} \cdot \left( T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\bar{h}_{ik}} \right) \right] \right\}^{\theta_2}. \end{aligned}$$

If parameters of the model, the level of productivity and the interest rate change then  $H_i$  and the equilibrium solution will change. It is important to note that  $\frac{\hat{B}_{ik}^*}{\bar{h}_{ik}}$  only depends on the parameters of the model (instead of being a function of lagged variables) and therefore does not introduce extra dynamics into the model. This is a consequence of assumption 2.

If  $h_0 < h_*$  human capital will be growing over time. The larger  $\theta_1$  is, the larger this growth will be. Galor and Tsiddon (1997) assume that  $\theta_1$  is a decreasing function of  $h$  so that one can have an increasing growth rate at low level of human capital ( $\theta_1 + \theta_2\Gamma_{i1} > 1$ ), and a decreasing rate at high levels.

Since  $\bar{G}_i$  is non-decreasing (when  $\bar{h}_i$  lower than the equilibrium level of human capital),  $\alpha_{Ei}$  will be non-increasing. If  $\bar{h}_i$  is low enough it will first decrease to  $\xi_1$  and thereafter remain constant.

There are two effects of a declining  $\alpha_{Ei}$  on the number of children,

$$\bar{n}_{ji} = \Omega_{ji} \left[ T_i^* + (1 + \theta_2) \frac{\hat{B}_{ik}^*}{\bar{h}_{ik}} \right],$$

because

$$\frac{\partial \Omega_{ji}}{\partial \alpha_{Ei}} > 0, \quad \frac{\partial \left( \hat{B}_{ik}^* / \bar{h}_{ik} \right)}{\partial \alpha_{Ei-1}} < 0, \quad \frac{\partial \left( \hat{B}_{ik}^* / \bar{h}_{ik} \right)}{\partial \alpha_{Ei-2}} < 0$$

as long as long as wealth accumulation and consumption have a greater weight in the utility function than children,  $1 + \beta + \beta^2 + \pi_{i+4,k}\beta^3 + \alpha_W > \alpha_0$ . The first order effect  $\frac{\partial \Omega_{ji}}{\partial \alpha_{Ei}} > 0$  will then always be positive. An increase in the desire for private old age care,  $\alpha_{Ei}$ , will increase the number of children as long as  $\alpha_0$  is not too large. If  $\alpha_0$  is very large, then one will wish to increase received care (per child), while slightly reducing the number of children. This will still lead to more total care.

An increase in  $\alpha_{Ei}$  makes private care relatively more important, increasing the desire for children, but, with a lag effect, the increase in the number of children and a corresponding decrease in wealth accumulation reduces bequests per child (leading to fewer children in later generations). Generally one would think that the direct effect would be stronger than the effect through bequests, so that over time a declining  $\alpha_{Ei}$  will lead to a decrease in the number of children (as has generally been observed).

As a decrease in  $\alpha_{Ei}$  decreases the number of children born (at least as a first order effect), the amount of education taken by women and men,  $\bar{e}_{if}$  and  $\bar{e}_{im}$  will increase. If in addition  $\varphi_{2if}^* > \varphi_{1if}^*$ , spacing and the average age when giving birth will increase (also fitting historical observations).

As soon as  $\alpha_{Ei}$  reaches a constant state then the number of children will also after a while become constant. The size of cohort  $i$  is given by

$$N_i = \frac{N_{i-1}}{2} \cdot \bar{n}_{1i-1} + \frac{N_{i-2}}{2} \cdot \bar{n}_{2i-2}.$$

The demographic dynamics can in general be described by

$$\begin{aligned} \mathbf{N}_{i+3} &= \begin{bmatrix} N_{i+2} & N_{i+1} \end{bmatrix} \cdot \begin{bmatrix} \frac{\bar{n}_{1i+2}}{2} & 1 \\ \frac{\bar{n}_{2i+1}}{2} & 0 \end{bmatrix} \\ &= \begin{bmatrix} N_{i+2} \cdot \bar{n}_{1i+2} + N_{i+1} \cdot \bar{n}_{2i+1} \\ N_{i+1} \end{bmatrix} \end{aligned}$$

If the number of children each cohort has is the same then the transition matrix becomes

$$\begin{bmatrix} \frac{\bar{n}_1}{2} & 1 \\ \frac{\bar{n}_2}{2} & 0 \end{bmatrix}.$$

The population will grow if  $\frac{\bar{n}_1}{2} + \frac{\bar{n}_2}{2} > 0$  and fall if  $\frac{\bar{n}_1}{2} + \frac{\bar{n}_2}{2} < 0$ , with  $\bar{n}_1 + \bar{n}_2 = 2$  leading to a stable population. If the starting values  $N_{-1}$  and  $N_{-2}$  are different, then the relative size of cohorts  $N_i/N_{i+1}$  will oscillate to begin with before approaching a constant value. The dominant eigenvalue of the transition matrix gives the natural growth rate of the population.

## 6 Changes in mortality, bargaining power of women and the cost of children

Decreasing mortality (increasing longevity  $\pi_{i+4,k}$ ) has two effects on the notional demand for children. The wish for more consumption in the last period will lead to fewer children, while the wish for more care will lead to more children.

**Proposition 3 (Falling mortality)** *Falling mortality (increased longevity  $\pi_{i+4}$ ) will increase the number of children born if*

$$1 + \beta + \beta^2 + \alpha_W > \left( \frac{\beta^3}{\alpha_{Ei}} + 1 \right) \alpha_0,$$

*otherwise falling mortality will decrease the number of children. A sufficiently low  $\alpha_{Ei}$  will lead to falling mortality decreasing the number of children born. If the number of children increases when mortality drops, the amount of education taken by women and men decrease, otherwise it will increase. If a decrease in mortality increases births and  $\varphi_{2if}^* > \varphi_{1if}^*$ , then an equal marginal increase in the longevity of women and men will lead to an decrease in spacing and in the average age at birth. If, on the other hand, a decrease in mortality decreases births, spacing and average age at birth will increase.*

If receiving care when old is not appreciated ( $\alpha_{Ei} = 0$ ), then a decrease in mortality (increase in longevity) leads to an increase in the desire for consumption at age 4 and thereby a decrease in the number of children. Individuals reallocate resources away from children toward consumption in the last period. On the other hand, if care when old is the only aspect of children one cares about ( $\alpha_0 = 0$ ), then a decrease in mortality (increase in longevity) makes receiving care when old more important, leading to an increase in the number of children. In this case individuals reallocate resources towards children.

Sufficient conditions for falling mortality to increase the number of children are that

$$\alpha_{Ei} > \beta^3$$

and that wealth accumulation and own consumption during one's working years have a greater weight in the utility function than having children,

$$1 + \beta + \beta^2 + \alpha_W > \alpha_0.$$

If early this century  $\alpha_{E_i}$  was large (due to a lack of government provided old age care), decreasing mortality lead in isolation to an increase in the number of children. To begin it could be argued that this effect was counteracted by the effect of increasing child costs (especially increases in the cost of older children,  $\varphi_{2i}$ ), so that the total effect was for the number of births to decrease. If the effect of increasing child costs abated, the effect of decreasing mortality would become relatively more important and births would start to increase. If then  $\alpha_{E_i}$  shifted downward as government provided old age care became more prevalent, the effect of decreasing mortality would be reversed, leading to a decrease in births.

**Proposition 4 (Increased bargaining power for women)** *If women have a smaller notional demand for children than men,  $\tilde{n}_{jif} < \tilde{n}_{jim}$  (which is always the case if  $\alpha_{E_i}$  is small enough,  $\varphi_{jf}^* > \varphi_{jm}^*$  and  $\pi_{i+4,f} > \pi_{i+4,m}$ ), then an increase in the bargaining power of women,  $\Phi_i$ , will decrease the number of children born. Such an increase will always lead to an increase in the average age of giving birth if*

$$\frac{(1 + r_{1+2})^{-1} (1 + r_{1+3})^{-1} w_{3ik}}{(1 + r_{1+2})^{-1} w_{2ik}} > \frac{(1 + r_{1+2})^{-1} w_{2ik}}{w_{1ik}}, \quad k = f, m.$$

*If an increase in  $\Phi_i$  decreases the demand for children, the amount of education taken by women and men,  $\bar{e}_{if}$  and  $\bar{e}_{im}$ , will increase.*

From this it follows that if  $\alpha_{E_i}$  becomes small enough for decreasing mortality to decrease the number of children, then increased bargaining power for women will also have a negative effect on the number of births, with positive effects on the average age at birth and educational levels.

The cost of having older children,  $\varphi_{2i}$ , can increase both due to falling economic benefits of having older children (they do less household work) or increasing costs (care becoming more expensive) of having older children.

**Proposition 5 (Increased costs of children)** *An increase in the cost of having older children,  $\varphi_{2i}$ , will lead to a fall in the number of births,  $\bar{n}_i$ . Such an increase will always lead to an increase in the average age of giving birth if*

$$\left( \frac{(1 + r_{1+2})^{-1} w_{2ik}}{w_{1ik}} \right)^2 > \frac{(1 + r_{1+2})^{-1} (1 + r_{1+3})^{-1} w_{3ik}}{(1 + r_{1+2})^{-1} w_{2ik}}, \quad k = f, m.$$

*An increase in  $\varphi_{2i}$  increases the amount of education taken by women,  $\bar{e}_{if}$ , and decreases the amount taken by men,  $\bar{e}_{im}$ . If womens' cost of having young children approaches that of men, then the difference in educational levels will decrease.*

It is only the relative time costs that are important when considering the amount of education taken (it plays no roll in the nominal demand functions, since the number of children one wants is proportional to the cost of taking care of these children). An increase in the cost of older children reduces the demand of men more than women. This implies that men's total time spent on children increases (the number of children they have is reduced by less than the cost of children) and that women's total time spent on children decreases (the number of children is reduced by more than the cost of children).

If the cost of having older children has been increasing and womens' cost of having young children approaches that of men, then womens' levels of education will increase more strongly than mens' levels (assuming the total effect is for education to increase).

In addition to the above results, the model is such that decrease in the importance of seniority through a reduction in the ratio  $\lambda_3/\lambda_1$  will lead to an increase in spacing (relatively more late births). A decrease in  $\lambda_3$  reduces  $\varphi_{2ik}^*$  while leaving  $\varphi_{1ik}^*$  unaffected, and an increase in  $\lambda_1$  increases  $\varphi_{1ik}^*$  while leaving  $\varphi_{2ik}^*$  unaffected. It becomes relatively cheaper to have children later (the opportunity cost has decreased).

The effect of falling mortality on desired levels of wealth and consumption (during working ages) are uncertain. The direct effect is negative since one wishes to increase old age care and consumption as old, but if the number of children decreases, the amount of education taken by women and men increases, leading to higher income and a positive income effect.

## 7 Conclusions

The paper has used a very simple combination of a overlapping generations model and bargaining among spouses to give a discussion of the factors that affect the number and timing of children. The model is able to explain changes in fertility as being the result of the net cost of having children, the utility of being cared for by ones children when one is old, and the bargaining strength of women in marriage.

The stylized facts discussed earlier can now be explained using the above model. Early in the century the benefits of having older children were decreasing fast due to fewer people living on farms in rural environments, leading to a decrease in the number of children. As this effect tapered off, declining mortality and a high utility of receiving care from one's children lead to the number of children rising again. Finally a fall in the utility of receiving care from one's children and increasing bargaining power among women lead to a further fall in fertility.

The increased bargaining power of women combined with less returns to seniority have recently lead to a delay of births. Education and the level of human capital have been increasing throughout

the last century. As a result of the increasing time cost of older children and of time costs of young children falling more rapidly for women than for men, the educational level of women has been approaching that of men.

## References

- [1] **Arroyo, C.R. and J. Zhang (1997):** ‘Dynamic Microeconomic Models of Fertility Choice: A Survey.’ *Journal of Population Economics* 10, 23-65.
- [2] **Arrondel, L. and A. Masson (2001):** ‘Family Transfers Involving Three Generations.’ *Scandinavian Journal of Economics* 103, 415-443.
- [3] **Aiyagari, S.R., J. Greenwood and N. Guner (2000):** ‘On the State of the Union.’ *Journal of Political Economy* 108, 213-244.
- [4] **Becker G.S., K.M. Murphy and R. Tamura (1990):** ‘Human Capital, Fertility and Economic Growth.’ *Journal of Political Economy* 98, S12-S37.
- [5] **Blackburn, K. and G.P. Cipriani (2002):** ‘A model of longevity, fertility and growth.’ *Journal of Economic Dynamics & Control* 26, 187–204.
- [6] **Fernández, R. (2002):** ‘Education, segregation and marital sorting: theory and an application to the UK’ *European Economic Review* 46, 993–1022.
- [7] **Galor, O. and D. Tsiddon (1997):** ‘The distribution of human capital and economic growth.’ *Journal of Economic Growth* 2, 93-124.
- [8] **Holz, V.J., J.A. Klerman and R.J. Willis (1997):** ‘The Economics of Fertility in Developed Countries’. In *Handbook of Population and Family Economics* edited by Rosenzweig and Stark, Elsevier Science B.V., Amsterdam: 275-347.
- [9] **Iyigun, M.F. (2000):** ‘Timing of childbearing and economic growth.’ *Journal of Development Economics* 61, 225-269.
- [10] **Jovanovic, B. (1979):** ‘Job Matching and the Theory of Turnover.’ *Journal of Political Economy* 87, 972-990.
- [11] **Mortensen, D.T. (1988):** ‘Matching: Finding a Partner for Life or Otherwise’. *American Journal of Sociology* 94 (suppl. 1988), 215-240.
- [12] **Yakita, A. (2001):** ‘Uncertain lifetime, fertility and social security.’ *Journal of Population Economics* 14, 625-640.



## Appendix A. Proof of Lemma 1 (Notional demands)

**Proof.** The optimization problem is

$$\begin{aligned} \max \quad & U_{ik} = \ln c_{1ik} + \beta \ln c_{2ik} + \beta^2 \ln c_{3ik} + \pi_{i+4,k} \beta^3 \ln c_{4ik} + \alpha_W \ln W_{ik} + \pi_{i+4,k} \cdot \alpha_E \ln E_{ik} \\ & + \frac{\alpha_0 + \pi_{i+4,k} \cdot \alpha_E i}{2} \ln n_{1ik} + \frac{\alpha_0 + \pi_{i+4,k} \cdot \alpha_E i}{2} \ln n_{2ik} + \pi_{i+4,k} \cdot \alpha_G \ln(1 + G_{ik}) \end{aligned}$$

$$\text{w.r.t } s_{1ik}, s_{2ik}, s_{3ik}, W_{ik}, E_{ik}, G_{ik}, n_{1ik}, n_{2ik}, e_{ik}$$

$$\begin{aligned} \text{s.t.} \quad & c_{1ik} = w_{1ik} (T - \Gamma_i \cdot e_{ik} - \varphi_{1ik} n_{1ik}) - s_{1ik} \\ & c_{2ik} = w_{2ik} (T - \varphi_{1ik} n_{2ik} - \varphi_{2i} n_{1ik}) - p_{E, i+2, k} \cdot \pi_{i+2}^* \eta_{2i} E_{ik} - p_{G, i+2} \cdot \pi_{i+2}^* \eta_{2i} G_{ik} \\ & \quad + (1 + r_{i+2}) s_{1ik} - s_{2ik} + \hat{B}_{i, i-2, k} \\ & c_{3ik} = w_{3ik} (T - \varphi_{2i} n_{2ik}) - p_{E, i+3, k} \cdot \pi_{i+3}^* \eta_{1i} E_{ik} - p_{G, i+3} \cdot \pi_{i+3}^* \eta_{1i} G_{ik} \\ & \quad + (1 + r_{i+3}) s_{2ik} - s_{3ik} + \hat{B}_{i, i-1, k} \\ & c_{4ik} = (1 + r_{i+4}) s_{3ik} - W_{ik} \end{aligned}$$

$$\begin{aligned} w_{1ik} &= \psi_{i+1} \bar{w}(r_{i+1}) \lambda_1 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} \\ w_{2ik} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} \\ w_{3ik} &= \psi_{i+3} \bar{w}(r_{i+3}) \lambda_3 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} \end{aligned}$$

Letting  $d_{kj} = (1 + r_k) \prod_{l=k}^j (1 + r_l)^{-1}$  maximization wrt  $s_{1ik}$ ,  $s_{2ik}$  and  $s_{3ik}$  leads to

$$\begin{aligned} c_{2ik} &= \frac{\beta}{d_{i+1, i+2}} c_{1ik} \\ c_{3ik} &= \frac{\beta^2}{d_{i+1, i+3}} c_{1ik} \\ c_{4ik} &= \frac{\pi_{i+4, k} \beta^3}{d_{i+1, i+4}} c_{1ik}. \end{aligned}$$

Inserting this into the the first order conditions for  $W_{ik}$ ,  $E_{ik}$ ,  $G_{ik}$ ,  $n_{1ik}$ ,  $n_{2ik}$ , and  $e_{ik}$ , and into the equation for total discounted consumption,

$$c_{1ik} + d_{i+1, i+2} c_{2ik} + d_{i+1, i+3} c_{3ik} + d_{i+1, i+4} c_{4ik} = (1 + \beta + \beta^2 + \pi_{i+4, k} \beta^3) c_{1ik},$$

gives the following six first order conditions

$$\begin{aligned} & (1 + \beta + \beta^2 + \pi_{i+4, k} \beta^3) c_{1ik} \\ & = (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} (T_i^* - \lambda_{1i}^* \Gamma_i e_{ik} - \varphi_{1ik}^* n_{1ik} - \varphi_{2ik}^* n_{2ik}) \\ & \quad - p_{E, i}^* E_{ik} - p_{G, i}^* G_{ik} - d_{i+1, i+4} W_{ik} + \hat{B}_{ik}^* \quad (\text{A.1}) \end{aligned}$$

$$n_{1ik} = \frac{\alpha_0 + \pi_{i+4, k} \cdot \alpha_E i}{(\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}} \frac{1}{2\varphi_{1ik}^*} c_{1ik} \quad (\text{A.2})$$

$$n_{2ik} = \frac{\alpha_0 + \pi_{i+4,k} \cdot \alpha_{Ei}}{(\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}} \frac{1}{2\varphi_{2ik}^*} c_{1ik} \quad (\text{A.3})$$

$$e_{ik} = \frac{1}{\lambda_{1i}^* \cdot \Gamma_i} \cdot \frac{\theta_2}{1 + \theta_2} [T_i^* - \varphi_{1ik}^* n_{1ik} - \varphi_{2i}^* n_{2ik}] \quad (\text{A.4})$$

$$E_{ik} = \frac{\pi_{i+4,k} \cdot \alpha_{Ei}}{p_{Ei}^*} c_{1ik} \quad (\text{A.5})$$

$$G_{ik} = \frac{\pi_{i+4,k} \cdot \alpha_G}{p_{Gi}^*} c_{1ik} \quad (\text{A.6})$$

$$W_{ik} = \frac{\alpha_W}{d_{i+1,i+4}} c_{1ik} \quad (\text{A.7})$$

in the seven variables  $c_{1ik}$ ,  $W_{ik}$ ,  $E_{ik}$ ,  $G_{ik}$ ,  $n_{1ik}$ ,  $n_{2ik}$ , and  $e_{ik}$ , where

$$\begin{aligned} \lambda_{1i}^* &= \psi_{i+1} \bar{w}(r_{i+1}) \lambda_1 \\ \lambda_{2i}^* &= d_{i+1,i+2} \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \\ \lambda_{3i}^* &= d_{i+1,i+3} \psi_{i+3} \bar{w}(r_{i+3}) \lambda_3 \\ \varphi_{1ik}^* &= \lambda_{1i}^* \varphi_{1ik} + \lambda_{2i}^* \varphi_{2i} \\ \varphi_{2ik}^* &= \lambda_{2i}^* \varphi_{1ik} + \lambda_{3i}^* \varphi_{2i} \\ T_i^* &= \lambda_{1i}^* \cdot T + \lambda_{2i}^* \cdot T + \lambda_{3i}^* \cdot T \\ p_{Ei}^* &= d_{i+1,i+2} p_{E,i+2} \cdot \pi_{i+2}^* \eta_{2i} + d_{i+1,i+3} p_{E,i+3} \cdot \pi_{i+3}^* \eta_{1i} \\ p_{Gi}^* &= d_{i+1,i+2} p_{G,i+2} \cdot \pi_{i+2}^* \eta_{2i} + d_{i+1,i+3} p_{G,i+3} \cdot \pi_{i+3}^* \eta_{1i} \\ \hat{B}_{ik}^* &= d_{i+1,i+2} \hat{B}_{i,i-2,k} + d_{i+1,i+3} \hat{B}_{i,i-1,k}. \end{aligned}$$

Defining  $R_{ik}$  as individual  $k$ 's total discounted income over his or her life,

$$R_{ik} = (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} (T_i^* - \lambda_{1i}^* \Gamma_i e_{ik}) + \hat{B}_{ik}^*,$$

with discounted expenditures equaling discounted income,

$$\begin{aligned} \sum_{j=1}^4 d_{i+1,i+j} \cdot c_{jik} + \tilde{h}_{ik} \cdot \varphi_{1ik}^* \cdot n_{1ik} + \tilde{h}_{ik} \cdot \varphi_{2ik}^* \cdot n_{2ik} + p_{Ei}^* \cdot E_{ik} + p_{Gi}^* \cdot G_{ik} + d_{i+1,i+4} \cdot W_{ik} \\ = R_{ik} = (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2} (T_i^* - \lambda_{1i}^* \Gamma_i e_{ik}) + \hat{B}_{ik}^*, \end{aligned}$$

the above seven first order equations can be solved wrt  $c_{1ik}$ ,  $W_{ik}$ ,  $E_{ik}$ ,  $G_{ik}$ ,  $n_{1ik}$ ,  $n_{2ik}$ , and  $e_{ik}$ , to get the notional demand equations

$$\begin{aligned} (\Lambda_{Nik} + \Lambda_{Eik}) \cdot \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \\ \tilde{c}_{1ik} = \frac{1}{\Lambda_{ik}^*} R_{ik} \\ \tilde{n}_{1ik} = \frac{1}{\varphi_{1ik}^*} \frac{\Lambda_{Nik} + \Lambda_{Eik}}{2} \frac{R_{ik}}{\tilde{h}_{ik}} \quad (\text{17}) \end{aligned}$$

$$\tilde{n}_{2ik} = \frac{1}{\varphi_{2ik}^*} \frac{\Lambda_{Nik} + \Lambda_{Eik}}{2} \frac{R_{ik}}{\tilde{h}_{ik}} \quad (18)$$

$$\tilde{e}_{ik} = \frac{\theta_2}{\lambda_{1i}^*} \cdot \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ T_i^* - \Lambda_{nik} \left( T_i^* + \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right) \right]$$

$$\tilde{E}_{ik} = \frac{1}{p_{Ei}^*} \Lambda_{Eik} R_{ik}$$

$$G_{ik} = \frac{1}{p_{Gi}^*} \Lambda_{Gik} R_{ik}$$

$$\tilde{W}_{ik} = \frac{1}{d_{i+1,i+4}} \Lambda_{W ik} R_{ik}$$

with

$$R_{ik} = \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ \tilde{h}_{ik} T_i^* + (1 + \theta_2) \hat{B}_{ik}^* \right]$$

where

$$\tilde{h}_{ik} = (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2}$$

and

$$\Lambda_{Nik} = \frac{1}{\Lambda_{ik}^*} \alpha_0, \quad \Lambda_{W ik} = \frac{1}{\Lambda_{ik}^*} \alpha_W,$$

$$\Lambda_{E ik} = \frac{1}{\Lambda_{ik}^*} (\pi_{i+4} \alpha_{Ei}),$$

$$\Lambda_{ik}^* = 1 + \beta + \beta^2 + \pi_{i+4,k} \beta^3 + \alpha_W + \pi_{i+4,k} \alpha_{Ei} + \alpha_0 + \pi_{i+4,k} \alpha_{Ei}.$$

■

## Appendix B. Proof of Lemma 4 (Effective demands)

**Proof.** Since children and government care for the elderly are given, we define the constant

$$u_i^* = (\alpha_0 + \pi_{i+4}^* \cdot \alpha_{Ei}) (\ln \bar{n}_{1i} + \ln \bar{n}_{2i}) + \pi_{i+4}^* \cdot \alpha_G \ln (1 + \bar{G}_{ik}),$$

so that the households maximization problem becomes:

$$\begin{aligned} \max \quad & \bar{U}_i = \ln c_{1i} + \beta \ln c_{2i} + \beta^2 \ln c_{3i} + \pi_{i+4}^* \beta^3 \ln c_{4i} + \alpha_W \ln W_i \\ & + \frac{1}{2} \pi_{i+4,f} \cdot \alpha_{Ei} \cdot \ln E_{if} + \frac{1}{2} \pi_{i+4,m} \cdot \alpha_{Ei} \cdot \ln E_{im} + u_i^* \\ \text{wrt} \quad & s_{1i}, s_{2i}, s_{3i}, e_{if}, e_{im}, W_i, E_{if}, E_{im} \\ \text{s.t.} \quad & c_{1i} = \frac{1}{2} [w_{1if} (T - \Gamma_i \cdot e_{if} - \varphi_{1if} \cdot \bar{n}_{1i}) + w_{1im} (T - \Gamma_i \cdot e_{im} - \varphi_{1im} \cdot \bar{n}_{1i}) - s_{1i} - 2\tau_{i+1}] \\ & c_{2i} = \frac{1}{2} [w_{2if} (T - \varphi_{1if} \cdot \bar{n}_{2i} - \varphi_{2i} \cdot \bar{n}_{1i}) + w_{2im} (T - \varphi_{1im} \cdot \bar{n}_{2i} - \varphi_{2i} \cdot \bar{n}_{1i}) \\ & \quad - \pi_{i+2}^* \eta_{2i} (p_{E,i+2,f} E_{if} + p_{E,i+2,m} E_{im}) + (1 + r_{i+2}) s_{1i} - s_{2i} + 2B_{i,i-2} - 2\tau_{i+2}] \\ & c_{3i} = \frac{1}{2} [(w_{3if} + w_{3im}) (T - \varphi_{2i} \cdot \bar{n}_{2i}) \\ & \quad - \pi_{i+3}^* \eta_{1i} (p_{E,i+3,f} E_{if} + p_{E,i+3,m} E_{im}) + (1 + r_{i+3}) s_{2i} - s_{3i} + 2B_{i,i-1} - 2\tau_{i+3}] \\ & c_{4i} = \frac{1}{2} (1 + r_{i+4}) s_{3i} - W_i \\ & \tau_i = \frac{p_{G_i}^* \cdot \pi_{i,f}^* N_i \cdot \bar{G}_i}{N_{i-3} + N_{i-2} + N_{i-1}} \end{aligned}$$

$$\begin{aligned} w_{1ik} &= \psi_{i+1} \bar{w}(r_{i+1}) \lambda_1 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, & k = f, m \\ w_{2ik} &= \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}, & k = f, m \\ w_{3ik} &= \psi_{i+3} \bar{w}(r_{i+3}) \lambda_3 \cdot (\bar{h}_{i-1})^{\theta_1} (e_{ik})^{\theta_2}. & k = f, m \end{aligned}$$

where savings  $s_{1i}, s_{2i}, s_{3i}$  are total savings for the household (all other variables refer to each person).

Inserting the first order conditions from maximization wrt  $s_{1i}, s_{2i}$  and  $s_{3i}$  into the first order conditions from from maximization wrt  $e_{1ik}$  and rearranging leads to the effective demands for education

$$\bar{e}_{if} = \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \cdot \Gamma_i} [T_i^* - \varphi_{1if}^* \cdot \bar{n}_{1i} - \varphi_{2if}^* \cdot \bar{n}_{2i}] \quad (22)$$

and

$$\bar{e}_{im} = \frac{\theta_2}{(1 + \theta_2) \lambda_{1i}^* \cdot \Gamma_i} [T_i^* - \varphi_{1im}^* \cdot \bar{n}_{1i} - \varphi_{2im}^* \cdot \bar{n}_{2i}], \quad (23)$$

where

$$\begin{aligned} \varphi_{1ik}^* &= \lambda_{1i}^* \cdot \varphi_{1k} + \lambda_{2i}^* \cdot \varphi_{2i} \\ \varphi_{2ik}^* &= \lambda_{2i}^* \cdot \varphi_{1k} + \lambda_{3i}^* \cdot \varphi_{2i} \end{aligned}$$

Inserting the first order conditions from maximization wrt  $s_{1i}$ ,  $s_{2i}$  and  $s_{3i}$  into the the first order conditions for  $W_{ik}$  and  $E_{ik}$  along with the equation for total discounted consumption,

$$c_{1i} + d_{i+1,i+2} \cdot c_{2i} + d_{i+1,i+3} \cdot c_{3i} + d_{i+1,i+4} \cdot c_{4i} = (1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3) c_{1i}$$

gives the following effective demands for old age care, wealth accumulation and consumption:

$$\bar{E}_{if} = \frac{1}{p_{Eif}^*} \frac{\pi_{i+4,f} \cdot \alpha_{Ei}}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.1})$$

$$\bar{E}_{im} = \frac{1}{p_{Eim}^*} \frac{\pi_{i+4,m} \cdot \alpha_{Ei}}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.2})$$

$$\bar{W}_i = \frac{1}{d_{i+1,i+4}} \frac{\alpha_W}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.3})$$

$$\bar{c}_{1i} = \frac{1}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.4})$$

$$\bar{c}_{2i} = \frac{1}{d_{i+1,i+2}} \frac{\beta}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.5})$$

$$\bar{c}_{3i} = \frac{1}{d_{i+1,i+3}} \frac{\beta^2}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.6})$$

$$\bar{c}_{4i} = \frac{1}{d_{i+1,i+4}} \frac{\pi_{i+4}^* \beta^3}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \quad (\text{B.7})$$

with total monetary income being

$$\frac{\bar{R}_i^M}{2} = \frac{1}{1 + \theta_2} \cdot \frac{h_{if} \bar{T}_{if} + h_{im} \bar{T}_{im}}{2} + B_i^* - \tau_i^*$$

where

$$\begin{aligned} \bar{T}_{if} &= T_i^* - \varphi_{1if}^* \cdot \bar{n}_{1i} - \varphi_{2if}^* \cdot \bar{n}_{2i} \\ \bar{T}_{im} &= T_i^* - \varphi_{1im}^* \cdot \bar{n}_{1i} - \varphi_{2im}^* \cdot \bar{n}_{2i} \\ \tilde{h}_{ik} &= (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2} \\ \bar{h}_i &= (\bar{h}_{i-1})^{\theta_1} \left( \frac{\bar{e}_{if} + \bar{e}_{im}}{2} \right)^{\theta_2} \end{aligned}$$

$$\begin{aligned}
\lambda_{1i}^* &= \psi_{i+1} \bar{w}(r_{i+1}) \lambda_1 \\
\lambda_{2i}^* &= d_{i+1,i+2} \psi_{i+2} \bar{w}(r_{i+2}) \lambda_2 \\
\lambda_{3i}^* &= d_{i+1,i+3} \psi_{i+3} \bar{w}(r_{i+3}) \lambda_3 \\
T_i^* &= \lambda_{1i}^* \cdot T + \lambda_{2i}^* \cdot T + \lambda_{3i}^* \cdot T \\
\varphi_{1ik}^* &= \lambda_{1i}^* \cdot \varphi_{1ik} + \lambda_{2i}^* \cdot \varphi_{2i} \\
\varphi_{2ik}^* &= \lambda_{2i}^* \cdot \varphi_{1ik} + \lambda_{3i}^* \cdot \varphi_{2i} \\
p_{Eik}^* &= d_{i+1,i+2} p_{E,i+2,k} \cdot \pi_{i+2}^* \eta_{2i} + d_{i+1,i+3} p_{E,i+3,k} \cdot \pi_{i+3}^* \eta_{1i} \\
p_{Gi}^* &= d_{i+1,i+2} p_{G,i+2} \cdot \pi_{i+2}^* \eta_{2i} + d_{i+1,i+3} p_{G,i+3} \cdot \pi_{i+3}^* \eta_{1i} \\
B_{ik}^* &= d_{i+1,i+2} \cdot B_{i,i-2,k} + d_{i+1,i+3} \cdot B_{i,i-1,k} \\
B_i^* &= d_{i+1,i+2} \cdot B_{i,i-2} + d_{i+1,i+3} \cdot B_{i,i-1} \\
\tau_i^* &= \tau_{i+1} + d_{i+1,i+2} \cdot \tau_{i+2} + d_{i+1,i+3} \cdot \tau_{i+3}. \\
R_{ik} &= \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ \tilde{h}_{ik} T_i^* + (1 + \theta_2) B_{ik}^* \right] \\
R_{ik}^M &= (1 - \Lambda_{Nik} - \Lambda_{Eik}) R_{ik} \\
\Phi_i &= \frac{\Phi_{if}^*}{\Phi_{if}^* + \Phi_{im}^*} \\
\Lambda_{Nik} &= \frac{\alpha_0}{1 + \beta + \beta^2 + \pi_{i+4} \beta^3 + \alpha_W + \alpha_0 + 2\pi_{i+4,k} \alpha_{Ei}} \\
\Lambda_{Eik} &= \frac{\pi_{i+4,k} \alpha_{Ei}}{1 + \beta + \beta^2 + \pi_{i+4} \beta^3 + \alpha_W + \alpha_0 + 2\pi_{i+4,k} \alpha_{Ei}}
\end{aligned}$$

■

## Appendix C. Proof of Lemma 5 (Actual bequests)

**Proof.** From Lemma 3 we have that

$$\bar{n}_{1i} = \Omega_{1i} \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right] \quad (19)$$

and

$$\bar{n}_{2i} = \Omega_{2i} \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right] \quad (20)$$

where

$$\begin{aligned}
\Omega_{1i} &= \frac{\Lambda_{Nif} + \Lambda_{Eif}}{2\varphi_{1if}^*} \cdot \frac{\Phi_i}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} + \frac{\Lambda_{Nim} + \Lambda_{Eim}}{2\varphi_{1im}^*} \cdot \frac{1 - \Phi_i}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})} \\
\Omega_{2i} &= \frac{\Lambda_{Nif} + \Lambda_{Eif}}{2\varphi_{2if}^*} \cdot \frac{\Phi_i}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} + \frac{\Lambda_{Nim} + \Lambda_{Eim}}{2\varphi_{2im}^*} \cdot \frac{1 - \Phi_i}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})}.
\end{aligned}$$

so that the total number of children born by generation  $i$  is

$$\bar{n}_{1i} + \bar{n}_{2i} = (\Omega_{1i} + \Omega_{2i}) \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} \right]$$

where

$$\begin{aligned} \Omega_{1i} + \Omega_{2i} = & \left( \frac{1}{\varphi_{2if}^*} + \frac{1}{\varphi_{1if}^*} \right) \frac{\Lambda_{Nif} + \Lambda_{Eif}}{2} \frac{\Phi_i}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} \\ & + \left( \frac{1}{\varphi_{2im}^*} + \frac{1}{\varphi_{1im}^*} \right) \frac{\Lambda_{Nim} + \Lambda_{Eim}}{2} \frac{(1 - \Phi_i)}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})}. \end{aligned}$$

We also earlier defined the notional income variables:

$$\begin{aligned} R_{ik} &= \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ \tilde{h}_{ik} T_i^* + (1 + \theta_2) \hat{B}_{ik}^* \right] \\ R_{ik}^M &= (1 - \Lambda_{Nik} - \Lambda_{Eik}) R_{ik} \end{aligned}$$

Bequests per child given by both parents can be written

$$\begin{aligned} B_i &= \frac{2\bar{W}_i + (1 - \pi_{i+4,f}) \bar{c}_{4i-1} + (1 - \pi_{i+4,m}) \bar{c}_{4i}}{\bar{n}_{1i} + \bar{n}_{2i}} \\ &= \frac{2\bar{W}_i + 2\bar{c}_{4i-1} - \left( \frac{\pi_{i+4,f} + \pi_{i+4,m}}{2} \right) 2\bar{c}_{4i}}{\bar{n}_{1i} + \bar{n}_{2i}} \\ &= \frac{2}{\bar{n}_{1i} + \bar{n}_{2i}} \left[ \bar{W}_i + (1 - \pi_{i+4}^*) \bar{c}_{4i} \right] \end{aligned}$$

Inserting from equations (B.3) and (B.7) in appendix B leads to

$$\begin{aligned} B_i = \frac{2}{\bar{n}_{1i} + \bar{n}_{2i}} & \left[ \frac{1}{d_{i+1,i+4}} \frac{\alpha_W}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \right. \\ & \left. + (1 - \pi_{i+4}^*) \frac{1}{d_{i+1,i+4}} \frac{\pi_{i+4}^* \beta^3}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \right] \end{aligned}$$

which can be written

$$\begin{aligned} B_i &= \frac{2}{\bar{n}_{1i} + \bar{n}_{2i}} \cdot \frac{1}{d_{i+1,i+4}} \frac{(1 - \pi_{i+4}^*) \pi_{i+4}^* \beta^3 + \alpha_W}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \cdot \frac{\bar{R}_i^M}{2} \\ &= \Omega_{0i} \cdot \frac{\bar{R}_i^M}{\bar{n}_{1i} + \bar{n}_{2i}} \end{aligned}$$

where

$$\frac{\bar{R}_i^M}{2} = \frac{1}{1 + \theta_2} \cdot \frac{h_{if} \bar{T}_{if} + h_{im} \bar{T}_{im}}{2} + B_i^* - \tau_i^*$$

and

$$\begin{aligned} \Omega_{0i} &= \frac{1}{d_{i+1,i+4}} \cdot \frac{(1 - \pi_{i+4}^*) \pi_{i+4}^* \beta^3 + \alpha_W}{1 + \beta + \beta^2 + \pi_{i+4}^* \beta^3 + \pi_{i+4}^* \alpha_{Ei} + \alpha_W} \\ &= \frac{1}{d_{i+1,i+4}} (\bar{\Lambda}_{Wi} + (1 - \pi_{i+4}^*) \bar{\Lambda}_{Ci}). \end{aligned}$$

$$\begin{aligned}
B_i &= \Omega_{0i} \cdot \frac{\bar{R}_i^M}{(\Omega_{1i} + \Omega_{2i}) \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{ik}^*}{h_{ik}} \right]} \\
&= \frac{\Omega_{0i}}{\Omega_{1i} + \Omega_{2i}} \cdot \frac{\bar{R}_i^M}{\frac{\left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{if}^*}{h_{if}} \right] + \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{im}^*}{h_{im}} \right]}{2}}
\end{aligned}$$

$$\begin{aligned}
&\frac{\left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{if}^*}{h_{if}} \right] + \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{im}^*}{h_{im}} \right]}{2} \\
&= \frac{1}{2} \left\{ \frac{1}{\tilde{h}_{if}} \left[ \tilde{h}_{if} T_i^* + (1 + \theta_2) \cdot \hat{B}_{if}^* \right] + \frac{1}{\tilde{h}_{im}} \left[ \tilde{h}_{im} T_i^* + (1 + \theta_2) \cdot \hat{B}_{im}^* \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&\frac{\left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{if}^*}{h_{if}} \right] + \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{im}^*}{h_{im}} \right]}{2} \\
&= \frac{1}{2} [1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})] [1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})] \\
&\cdot \left\{ \frac{1}{\tilde{h}_{if}} \cdot \frac{\left[ \tilde{h}_{if} T_i^* + (1 + \theta_2) \cdot \hat{B}_{if}^* \right]}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} \cdot \frac{1}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})} \right. \\
&\quad \left. + \frac{1}{\tilde{h}_{im}} \cdot \frac{\left[ \tilde{h}_{im} T_i^* + (1 + \theta_2) \cdot \hat{B}_{im}^* \right]}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})} \cdot \frac{1}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} \right\}.
\end{aligned}$$

Inserting from

$$R_{ik} = \frac{1}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ \tilde{h}_{ik} T_i^* + (1 + \theta_2) \hat{B}_{ik}^* \right]$$

leads to

$$\begin{aligned}
&\frac{\left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{if}^*}{h_{if}} \right] + \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{im}^*}{h_{im}} \right]}{2} \\
&= \frac{1}{2} [1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})] [1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})] \\
&\quad \cdot \left\{ \frac{1}{\tilde{h}_{if}} R_{if} \frac{1}{1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})} \right. \\
&\quad \left. + \frac{1}{\tilde{h}_{im}} R_{im} \frac{1}{1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})} \right\}
\end{aligned}$$

$$\begin{aligned}
&\frac{\left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{if}^*}{h_{if}} \right] + \left[ T_i^* + (1 + \theta_2) \cdot \frac{\hat{B}_{im}^*}{h_{im}} \right]}{2} \\
&= \frac{1}{\tilde{h}_{if}} R_{if} \frac{[1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})]}{2} + \frac{1}{\tilde{h}_{im}} R_{im} \frac{[1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})]}{2}
\end{aligned}$$



so that

$$B_i = \frac{\Omega_{0i}}{\Omega_{1i} + \Omega_{2i}} \cdot \bar{R}_i^M \cdot \left( \frac{1}{\tilde{h}_{if}} R_{if}^M \frac{[1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})]}{2} + \frac{1}{\tilde{h}_{im}} R_{im}^M \frac{[1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})]}{2} \right)^{-1}$$

Inserting from

$$\begin{aligned} R_{ik}^M &= (1 - \Lambda_{Nik} - \Lambda_{Eik}) R_{ik} \\ R_{ik} &= \frac{1}{1 - \Lambda_{Nik} - \Lambda_{Eik}} R_{ik}^M \end{aligned}$$

leads to

$$B_i = \frac{\Omega_{0i}}{\Omega_{1i} + \Omega_{2i}} \cdot \bar{R}_i^M \cdot \left( \frac{1}{\tilde{h}_{if}} R_{if}^M \frac{[1 + \theta_2 (1 - \Lambda_{Nif} - \Lambda_{Eif})]}{2(1 - \Lambda_{Nif} - \Lambda_{Eif})} + \frac{1}{\tilde{h}_{im}} R_{im}^M \frac{[1 + \theta_2 (1 - \Lambda_{Nim} - \Lambda_{Eim})]}{2(1 - \Lambda_{Nim} - \Lambda_{Eim})} \right)^{-1}$$

$$\begin{aligned} B_i &= \frac{\Omega_{0i}}{\Omega_{1i} + \Omega_{2i}} \cdot \bar{R}_i^M \cdot \left( \frac{1}{\tilde{h}_{if}} R_{if}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{Nif} - \Lambda_{Eif}} + \theta_2 \right) \right. \\ &\quad \left. + \frac{1}{\tilde{h}_{im}} R_{im}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{Nim} - \Lambda_{Eim}} + \theta_2 \right) \right)^{-1} \end{aligned}$$

This expression must also apply to earlier generations:

$$\begin{aligned} B_{i-j} &= \frac{\Omega_{0,i-j}}{\Omega_{1,i-j} + \Omega_{2,i-j}} \cdot \bar{R}_{i-j}^M \cdot \left( \frac{1}{\tilde{h}_{i-j,f}} R_{i-j,f}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f}} + \theta_2 \right) \right. \\ &\quad \left. + \frac{1}{\tilde{h}_{i-j,m}} R_{i-j,m}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m}} + \theta_2 \right) \right)^{-1} \end{aligned}$$

Finally this implies that

$$\begin{aligned} B_{i,i-j} &= \eta_{ji} \cdot B_{i-j} \\ &= \eta_{ji} \frac{\Omega_{0,i-j}}{\Omega_{1,i-j} + \Omega_{2,i-j}} \cdot \bar{R}_{i-j}^M \cdot \left( \frac{1}{\tilde{h}_{i-j,f}} R_{i-j,f}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f}} + \theta_2 \right) \right. \\ &\quad \left. + \frac{1}{\tilde{h}_{i-j,m}} R_{i-j,m}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m}} + \theta_2 \right) \right)^{-1} \quad (24) \end{aligned}$$

where

$$\begin{aligned} \Omega_{0,i-j} &= \frac{1}{d_{i-j+1,i-j+4}} \cdot \frac{(1 - \pi_{i-j+4}^*) \pi_{i-j+4}^* \beta^3 + \alpha_W}{1 + \beta + \beta^2 + \pi_{i-j+4}^* \beta^3 + \pi_{i-j+4}^* \alpha_E i + \alpha_W} \\ \Omega_{1,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{1,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{\Lambda_{N,i-j,m} + \Lambda_{E,i-j,m}}{2\varphi_{1,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})} \\ \Omega_{2,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{2,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{1}{2\varphi_{2,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})} \end{aligned}$$

■

## Appendix D. Proof of Lemma 2 (Expected bequests)

**Proof.** Lemma 5 (appendix C) states that individuals in generation  $i - j$  leave behind actual bequests to generation  $i$  of  $B_{i,i-j}$ ,

$$\begin{aligned} B_{i,i-j} &= \eta_{ji} \cdot B_{i-j} \\ &= \eta_{ji} \frac{\Omega_{0,i-j}}{\Omega_{1,i-j} + \Omega_{2,i-j}} \cdot \bar{R}_{i-j}^M \cdot \left( \frac{1}{\tilde{h}_{i-j,f}} R_{i-j,f}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f}} + \theta_2 \right) \right. \\ &\quad \left. + \frac{1}{\tilde{h}_{i-j,m}} R_{i-j,m}^M \cdot \frac{1}{2} \left( \frac{1}{1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m}} + \theta_2 \right) \right)^{-1} \end{aligned} \quad (24)$$

where

$$\begin{aligned} \tilde{h}_{ik} &= (\bar{h}_{i-1})^{\theta_1} (\tilde{e}_{ik})^{\theta_2} \\ R_{ik}^M &= \frac{1 - \Lambda_{Nik} - \Lambda_{Eik}}{1 + \theta_2 (1 - \Lambda_{Nik} - \Lambda_{Eik})} \left[ \tilde{h}_{ik} T_i^* + (1 + \theta_2) \hat{B}_{ik}^* \right] \\ \bar{R}_i^M &= h_{if} \left[ T_i^* - \lambda_{1i}^* \cdot \Gamma_i \cdot e_{if} - \varphi_{1if}^* \cdot \bar{n}_{1i} - \varphi_{2if}^* \cdot \bar{n}_{2i} + \frac{B_i^* - \tau_i^*}{\tilde{h}_{if}} \right] \\ &\quad + h_{im} \left[ T_i^* - \lambda_{1i}^* \cdot \Gamma_i \cdot e_{im} - \varphi_{1im}^* \cdot \bar{n}_{1i} - \varphi_{2im}^* \cdot \bar{n}_{2i} + \frac{B_{im}^* - \tau_i^*}{\tilde{h}_{im}} \right] \end{aligned}$$

and

$$\begin{aligned} \Omega_{0,i-j} &= \frac{1}{d_{i-j+1,i-j+4}} \cdot \frac{(1 - \pi_{i-j+4}^*) \pi_{i-j+4}^* \beta^3 + \alpha_W}{1 + \beta + \beta^2 + \pi_{i-j+4}^* \beta^3 + \pi_{i-j+4}^* \alpha_{Ei} + \alpha_W} \\ \Omega_{1,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{1,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{\Lambda_{N,i-j,m} + \Lambda_{E,i-j,m}}{2\varphi_{1,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})} \\ \Omega_{2,i-j} &= \frac{\Lambda_{N,i-j,f} + \Lambda_{E,i-j,f}}{2\varphi_{2,i-j,f}^*} \cdot \frac{\Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})} \\ &\quad + \frac{1}{2\varphi_{2,i-j,m}^*} \cdot \frac{1 - \Phi_{i-j}}{1 + \theta_2 (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})} \end{aligned}$$

It is then easily seen that assumption 2 implies that by replacing  $h_{i-j,f}$  and  $h_{i-j,m}$  with  $\tilde{h}_{ik}$ , and  $R_{i-j,k}^M$  with  $\frac{\hat{R}_{i-j}^M}{2}$ , the received bequests from generation  $i - j$ ,  $\hat{B}_{i,i-j,k}$ , can be written

$$\hat{B}_{i,i-j,k} = 2\eta_{ji} \frac{\Omega_{0,i-j}}{\Omega_{1,i-j} + \Omega_{2,i-j}} \cdot \tilde{h}_{ik} \cdot \left( \frac{(1 - \Lambda_{N,i-j,f} - \Lambda_{E,i-j,f})^{-1} + (1 - \Lambda_{N,i-j,m} - \Lambda_{E,i-j,m})^{-1}}{2} + \theta_2 \right)^{-1}$$

Total discounted bequests per unit of human capital then becomes

$$\frac{\hat{B}_{ik}^*}{\tilde{h}_{ik}} = (1 + r_{i+2})^{-1} \frac{\hat{B}_{i,i-2,k}}{\tilde{h}_{ik}} + (1 + r_{i+2})^{-1} (1 + r_{i+3})^{-1} \frac{\hat{B}_{i,i-1,k}}{\tilde{h}_{ik}}$$

where

$$\frac{\hat{B}_{i,i-1,k}}{\tilde{h}_{ik}} = 2\eta_{1i} \cdot \frac{\Omega_{0,i-1}}{\Omega_{1,i-1} + \Omega_{2,i-1}} \cdot \left( \frac{(1 - \Lambda_{N,i-1,f} - \Lambda_{E,i-1,f})^{-1} + (1 - \Lambda_{N,i-1,m} - \Lambda_{E,i-1,m})^{-1}}{2} + \theta_2 \right)^{-1}$$

and

$$\frac{\hat{B}_{i,i-2,k}}{\tilde{h}_{ik}} = 2\eta_{2i} \cdot \frac{\Omega_{0,i-2}}{\Omega_{1,i-2} + \Omega_{2,i-2}} \cdot \left( \frac{(1 - \Lambda_{N,i-2,f} - \Lambda_{E,i-2,f})^{-1} + (1 - \Lambda_{N,i-2,m} - \Lambda_{E,i-2,m})^{-1}}{2} + \theta_2 \right)^{-1}$$

with

$$\hat{B}_{i-j,f}^* / \tilde{h}_{if} = \hat{B}_{i-j,m}^* / \tilde{h}_{im}, \quad j = 1, 2.$$

Generation  $i$ 's expected received bequests divided by their notional human capital level,  $\frac{\hat{B}_{ik}^*}{(\bar{h}_{i-1})^{\theta_1} (\bar{e}_{ik})^{\theta_2}}$ , is thereby given in equation (B.1), only depending on model parameters  $\theta_2, \lambda_1, \lambda_2, \lambda_3, \psi_{i+1}, \psi_{i+2}, \psi_{i+3}, r_{i+1}, r_{i+2}, r_{i+3}, \varphi_{1i-2f}, \varphi_{1i-1f}, \varphi_{1if}, \varphi_{1i-2m}, \varphi_{1i-1m}, \varphi_{1im}, \varphi_{2i-2}, \varphi_{2i-1}, \varphi_{2i}, \pi_{i+2,f}, \pi_{i+3,f}, \pi_{i+4,f}, \pi_{i+2,m}, \pi_{i+3,m}, \pi_{i+4,m}, \Phi_{i-2}, \Phi_{i-1}, \Phi_i, \beta, \alpha_W, \alpha_{Ei-2}, \alpha_{Ei-1}, \alpha_{Ei}$ , and  $\alpha_0$ . ■